Vector-valued p-adic Measures

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1 Preliminaries

Throughout this paper, $\mathbb{K}$ stands for a complete non-Archimedean valued field whose valuation is non-trivial. By a seminorm, on a vector space E over $\mathbb{K}$, we mean a non-Archimedean seminorm. Also by a locally convex space we will mean a non-Archimedean locally convex space over $\mathbb{K}$ (see [12] and [13]). For E a locally convex space, we denote by $cs(E)$ the collection of all continuous seminorms on E and by E' the topological dual of E. For a zero-dimensional Hausdorff topological space X, $\beta_o X$ is the Banachewski compactification of X, $C_b(X)$ the space of all continuous $\mathbb{K}$-valued functions on X and $C_{rc}(X)$ the space of all $f \in C_b(X)$ whose range is relatively compact. Every $f \in C_{rc}(X)$ has a continuous extension f^{β_o} to all of $\beta_o X$. For $f \in \mathbb{K}^X$ and $A \subset X$, we define

$$\|f\|_A = \sup\{|f(x)| : x \in A\} \quad and \quad \|f\| = \|f\|_X.$$

By $\overline{A}^{\beta_o X}$ we will denote the closure of A in $\beta_o X$.
Next we will recall the definition of the strict topology β on $C_b(X)$ which was given in [5]. Let Ω be the family of all compact subsets of $\beta_o X$ which are disjoint from X. For $Z \in \Omega$, let C_Z be the set of all $h \in C_{rc}(X)$ for which h^{β_o} vanishes on Z. We denote by β_Z the locally convex topology on $C_b(X)$ generated by the seminorms p_h, $h \in C_Z$, where $p_h(f) = \|hf\|$. The inductive limit of the topologies β_Z, $Z \in \Omega$, is the strict topology β. As it is shown in [7], Theorem 2.2, an absolutely convex subset W of $C_b(X)$ is a β_Z-neighborhood of zero iff, for each $r > 0$, there exist a clopen subset A of X, with $\overline{A}^{\beta_o X}$ disjoint from Z, and $\epsilon > 0$ such that

$$\{f \in C_b(X) : \|f\|_A \le \epsilon,\ \|f\| \le r\} \subset W.$$

Monna and Springer initiated in [11] non-Archimedean integration. In [13] and [14], van Rooij and Schikhof developed a non-Archimedean integration theory for scalar valued measures. Some results on measures with values in Banach spaces were given in [1], [2] and [3]. In this paper we will study measures with values in a locally convex space as well as integrals of scalar valued functions with respect to such measures.

2 Vector Measures

Let $\mathcal{R}$ be a separating algebra of subsets of a non-empty set X, i.e. $\mathcal{R}$ is a family of subsets of X with the following properties :

1. $X \in \mathcal{R}$ and, if $A, B \in \mathcal{R}$, then $A \cup B$, $A \cap B$, $A \setminus B$ are also in $\mathcal{R}$.
2. If x, y are distinct elements of X, then there exists a member of $\mathcal{R}$ containing x but not y.

We will call the members of $\mathcal{R}$ measurable sets. Clearly $\mathcal{R}$ is a base for a Hausdorff zero-dimensional topology $\tau_{\mathcal{R}}$ on X.
For a net (V_δ) of subsets of X we will write $V_\delta \downarrow \emptyset$ if it is decreasing and $\bigcap V_\delta = \emptyset$. Similarly we will write $V_n \downarrow \emptyset$ for a sequence (V_n) of sets which decreases to the empty set.
Let now E be a Hausdorff locally convex space. We denote by $M(\mathcal{R}, E)$ the space of all bounded finitely-additive measures $m : \mathcal{R} \to E$. For $m \in M(\mathcal{R}, E)$ and $p \in cs(E)$, we define

$$m_p : \mathcal{R} \to \mathbf{R}, \quad m_p(A) = \sup\{p(m(V)) : V \in \mathcal{R}, V \subset A\}$$

and $\|m\|_p = m_p(X)$. We also define

$$N_{m,p} : X \to \mathbf{R}, \quad N_{m,p}(x) = \inf\{m_p(V) : x \in V \in \mathcal{R}\}.$$

An element m of $M(\mathcal{R}, E)$ is called :

1. σ-additive if $m(V_n) \to 0$ if $V_n \downarrow \emptyset$.
2. τ-additive if $m(V_\delta) \to 0$ if $V_\delta \downarrow \emptyset$.

Let $M_\sigma(\mathcal{R}, E)$ (resp. $M_\tau(\mathcal{R}, E)$) be the space of all σ-additive (resp. τ-additive) members of $M(\mathcal{R}, E)$.

Theorem 2.1 *Let $m \in M(\mathcal{R}, E)$. Then*

1. *m is τ-additive iff, for all $p \in cs(E)$, we have that $m_p(V_\delta) \to 0$ when $V_\delta \downarrow \emptyset$.*
2. *m is σ-additive iff, for all $p \in cs(E)$, we have that $m_p(V_n) \to 0$ when $V_n \downarrow \emptyset$.*

Proof : (1). Clearly the condition is sufficient. Conversely, assume that m is τ-additive but the condition is not satisfied. Then there exist a $p \in cs(E)$, an $\epsilon > 0$ and a net $(V_\delta)_{\delta \in \Delta}$ of measurable sets which decreases to the empty set such that $m_p(V_\delta) > \epsilon$ for all δ.
Claim : For each $\delta \in \Delta$, there exist $\gamma \geq \delta$ and a measurable set A such that $V_\gamma \subset A \subset V_\delta$ and $p(m(A)) > \epsilon$. Indeed, there exists $B \subset V_\delta$ with $p(m(B)) > \epsilon$. For each $\gamma \in \Delta$, set $Z_\gamma = B \cap V_\gamma$, $W_\gamma = V_\gamma \setminus Z_\gamma$. Then $W_\gamma \downarrow \emptyset$. Since m is τ-additive, there exists $\gamma \geq \delta$ such that $p(m(W_\gamma)) < \epsilon$. The sets B and W_γ are disjoint. If $A = W_\gamma \cup B$, then $V_\gamma \subset A \subset V_\delta$ and

$$p(m(A)) = p(m(W_\gamma) + m(B)) = p(m(B)) > \epsilon,$$

which proves the claim.
Let now $\mathcal{F}$ be the family of all measurable sets A such that there are $\gamma \geq \delta$ with $V_\gamma \subset A \subset V_\delta$ and $p(m(A)) > \epsilon$. Since $\mathcal{F} \downarrow \emptyset$, we arrived at a contradiction. This proves (1).
(2). The proof is analogous to that of (1).

Theorem 2.2 *Let $m \in M_\tau(\mathcal{R}, E)$ and let $(V_i)_{i\in I}$ be a family of measurable sets. If $p \in cs(E)$, then for each measurable subset V of $\bigcup_{i\in I} V_i$, we have that*

$$m_p(V) \leq \sup_i m_p(V_i).$$

Proof : For each finite subset S of I, let $W_S = \bigcup_{i\in S} V_i$. Then $V \bigcap W_S^c \downarrow \emptyset$. If $m_p(V) > 0$, there exists a finite subset S of I such that $m_p(V \bigcap W_S^c) < m_p(V)$. Now

$$\begin{aligned} m_p(V) &= \max\{m_p(V \cap W_S), \quad m_p(V \cap W_S^c)\} \\ &= m_p(V \cap W_S) \leq m_p(W_S) = \max_{i\in S} m_p(V_i). \end{aligned}$$

Corollary 2.3 *Let $m \in M_\tau(\mathcal{R}, E)$, $p \in cs(E)$ and $V \in \mathcal{R}$. Then*

$$m_p(V) = \sup_{x\in V} N_{m,p}(x).$$

Proof : Clearly $m_p(V) \geq \alpha = \sup_{x\in V} N_{m,p}(x)$. On the other hand, if $\epsilon > 0$, then for each $x \in V$ there exists a measurable set V_x, with $x \in V_x \subset V$, such that $m_p(V_x) < N_{m,p}(x) + \epsilon \leq \alpha + \epsilon$. Since $V = \bigcup_{x\in V} V_x$, we have that

$$m_p(V) \leq \sup_{x\in V} m_p(V_x) \leq \alpha + \epsilon,$$

and the result follows as $\epsilon > 0$ was arbitrary.

Theorem 2.4 *Let $m \in M_\sigma(\mathcal{R}, E)$ and let (V_n) be a sequence of measurable sets. If $V \in \mathcal{R}$ is contained in $\bigcup V_n$, then $m_p(V) \leq \sup_n m_p(V_n)$.*

Proof : Let $W_n = \bigcup_{k=1}^n V_k$. Suppose that $m_p(V) > 0$. Since $V \bigcap W_n^c \downarrow \emptyset$, there exists an n such that $m_p(V \cap W_n^c) < m_p(V)$. Now

$$\begin{aligned} m_p(V) &= \max\{m_p(V \cap W_n^c), m_p(V \cap W_n)\} \\ &= m_p(V \cap W_n) \leq m_p(W_n) = \max_{1\leq k\leq n} m_p(V_k). \end{aligned}$$

Theorem 2.5 *If $m \in M(\mathcal{R}, E)$ and $p \in cs(E)$, then $N_{m,p}$ is upper semicontinuous.*

Proof: Let $\alpha > 0$ and $V = \{x : N_{m,p}(x) < \alpha\}$. For $x \in V$, there exists a measurable set A containing x and such that $m_p(A) < \alpha$. Now $x \in A \subset V$ and so V is open.

Theorem 2.6 *Let $m \in M_\tau(\mathcal{R}, E)$, $p \in cs(E)$ and $\epsilon > 0$. Then the set*

$$X_{p,\epsilon} = \{x : N_{m,p}(x) \geq \epsilon\}$$

is $\tau_{\mathcal{R}}$-compact.

Proof : Let $(V_i)_{i\in I}$ be a family of measurable sets covering $X_{p,\epsilon} = Y$. Since $N_{m,p}$ is upper semicontinuous, the set Y is closed. For each finite subset S of I, let $W_S = \bigcup_{i\in S} V_i$. Consider the family $\mathcal{F}$ of all measurable sets of the form $[W_S \cup V]^c$, where V is a measurable set disjoint from Y and S a finite subset of I. Then $\mathcal{F}$ is downwards directed and $\bigcap \mathcal{F} = \emptyset$. Since m is τ-additive, there are S and V such that $m_p([W_S \cup V]^c) < \epsilon$. But then $[W_S \cup V]^c \subset Y^c$, and thus $Y \subset W_S \cup V$, which implies that $Y \subset W_S$. This completes the proof.

Definition 2.7 *A subset G of X is said to be a support set of an $m \in M(\mathcal{R}, E)$ if $m(V) = 0$ for each measurable set V disjoint from G.*

Theorem 2.8 *Let $m \in M_\tau(\mathcal{R}, E)$. Then the set*

$$supp(m) = \overline{\bigcup_{p\in cs(E)} \{x : N_{m,p}(x) > 0\}}$$

is the smallest of all closed support sets of m.

Proof : If V is a measurable set disjoint from $supp(m)$, then for each $p \in cs(E)$ we have

$$p(m(V)) \leq m_p(V) = \sup_{x\in V} N_{m,p}(x) = 0,$$

which proves that $supp(m)$ is a support set of m since E is Hausdorff. On the other hand, let F be a closed support set of m. Given $x \in F^c$, there exists $V \in \mathcal{R}$ with $x \in V \subset F^c$. Now, for each $p \in cs(E)$ and $y \in V$, we have that $N_{m,p}(y) \leq m_p V) = 0$ and so the set

$$B = \bigcup_{p\in cs(E)} \{x : N_{m,p}(x) \neq 0\}$$

does not intersect V, which implies that $x \notin \overline{B} = supp(m)$. Thus $supp(m) \subset F$ and the result follows.

3 A Universal Measure

Let $\mathcal{R}$ be a separating algebra of subsets of X and let $S(\mathcal{R})$ be the vector space of all $\mathbb{K}$-valued $\mathcal{R}$-simple functions on X. Let

$$\chi : \mathcal{R} \to S(\mathcal{R}), \quad A \mapsto \chi_A.$$

Let E be a Hausdorff locally convex space. Every $m \in M(\mathcal{R}, E)$ induces a linear map

$$\hat{m} : S(\mathcal{R}) \to E, \quad \hat{m}\left(\sum_{k=1}^n \lambda_k \chi_{V_k}\right) = \sum_{k=1}^n \lambda_k m(V_k).$$

On $S(\mathcal{R})$ we consider the locally convex topologies ϕ, ϕ_σ, ϕ_τ defined as follows :

1. ϕ is the weakest locally convex topology for which, for each Hausdorff locally convex space E and each $m \in M(\mathcal{R}, E)$, the map $\hat{m} : S(\mathcal{R}) \to E$ is continuous.

2. ϕ_σ is the weakest locally convex topology for which, for each Hausdorff locally convex space E and each $m \in M_\sigma(\mathcal{R}, E)$, the map $\hat{m} : S(\mathcal{R}) \to E$ is continuous.

3. ϕ_τ is the weakest locally convex topology for which, for each Hausdorff locally convex space E and each $m \in M_\tau(\mathcal{R}, E)$, the map $\hat{m} : S(\mathcal{R}) \to E$ is continuous.

Clearly $\phi_\tau \subset \phi_\sigma \subset \phi$.

Lemma 3.1 *The topology ϕ_τ is Hausdorff.*

Proof: Every $x \in X$ defines a τ-additive measure

$$m_x : \mathcal{R} \to \mathbb{K}, \quad m_x(A) = \chi_A(x).$$

Let $g \in S(\mathcal{R})$, $g \neq 0$ and let $g(x) \neq 0$. Let $0 < \epsilon < |g(x)|$. The set

$$\{h \in S(\mathcal{R}) : |\hat{m}_x(h)| = |h(x)| < \epsilon\}$$

is a ϕ_τ-neighborhood of zero not containing g.

Theorem 3.2 *If $F = (S(\mathcal{R}), \rho)$, where $\rho = \phi$, ϕ_σ or ϕ_τ, then $\chi : \mathcal{R} \to F$ is a member of $M(\mathcal{R}, F)$, $M_\sigma(\mathcal{R}, F)$ or $M_\tau(\mathcal{R}, F)$, respectively.*

Proof : Assume that $F = (S(\mathcal{R}), \phi_\tau)$. Clearly χ is finitely additive. Let E be a Hausdorff locally convex space and let $m \in M_\tau(\mathcal{R}, E)$, $p \in cs(E)$. Let

$$W = \{s \in E : p(s) \leq 1\}.$$

Since $m \in M_\tau(\mathcal{R}, E)$, there exists $\lambda \in \mathbb{K}$ such that $m(\mathcal{R}) \subset \lambda W$. If

$$D = \{g \in S(\mathcal{R}) : \hat{m}(g) \in W\},$$

then $\chi(\mathcal{R}) \subset \lambda D$, which proves that $\chi : \mathcal{R} \to F$ is bounded. If (V_δ) is a net of measurable sets with $V_\delta \downarrow \emptyset$, then $m(V_\delta) \to 0$, and so $m(V_\delta) \in W$ eventually, which implies that $\chi_{V_\delta} \in D$ eventually. Thus $\chi \in M_\tau(\mathcal{R}, F)$. The proofs for the cases of ϕ and ϕ_σ are analogous.

Theorem 3.3 *Let E be a Hausdorff locally convex space. Then :*

1. *The map $m \mapsto \hat{m}$, from $M(\mathcal{R}, E)$ to the space $L((S(\mathcal{R}), \phi), E)$, of all continuous linear maps from $(S(\mathcal{R}), \phi)$ to E, is an algebraic isomorphism.*

2. *The map $m \mapsto \hat{m}$, from $M_\sigma(\mathcal{R}, E)$ to the space $L((S(\mathcal{R}), \phi_\sigma), E)$, is an algebraic isomorphism.*

3. *The map $m \mapsto \hat{m}$, from $M_\tau(\mathcal{R}, E)$ to the space $L((S(\mathcal{R}), \phi_\tau), E)$, is an algebraic isomorphism.*

Proof : (1) By the definition of ϕ, each $\hat{m}$ is continuous. On the other hand, let $u : (S(\mathcal{R},), \phi) \to E$ be a continuous linear map and take $m = u \circ \chi$. Then $m \in M(\mathcal{R}, E)$ and $\hat{m} = u$. The proofs of (2) and (3) are analogous.

Since, for every Hausdorff locally convex space E, every measure $m : \mathcal{R} \to E$ is of the form $m = u \circ \chi$, for some ϕ-continuous linear map u from $S(\mathcal{R})$ to E, we will refer to the measure $\chi : \mathcal{R} \to (S(\mathcal{R}), \phi)$ as a universal measure. Taking $\mathbb{K}$ in place of E and identifying each scalar measure μ on $\mathcal{R}$ by the corresponding linear functional $\hat{\mu}$, we get the following

Theorem 3.4 *The spaces $M(\mathcal{R}) = M(\mathcal{R}, \mathbb{K})$, $M_\sigma(\mathcal{R})$ and $M_\tau(\mathcal{R})$ are algebraically isomorphic with the spaces $(S(\mathcal{R}), \phi)'$, $(S(\mathcal{R}), \phi_\sigma)'$ and $(S(\mathcal{R}), \phi_\tau)'$, respectively.*

Theorem 3.5 *On the space $S(\mathcal{R})$, the topology ϕ is coarser than the topology τ_u of uniform convergence.*

Proof : Let E be a Hausdorff locally convex space and let $m \in M(\mathcal{R}, E)$. It suffices to show that $\hat{m} : (S(\mathcal{R}), \tau_u) \to E$ is continuous. Indeed, let $p \in cs(E)$. There exists $r > 0$ such that $p(m(A)) \leq r$ for all $A \in \mathcal{R}$. Now, for

$$V = \{g \in S(\mathcal{R}) : \|g\| \leq 1/r\},$$

we have that $p(\hat{m}(g)) \leq 1$ for all $g \in V$. Indeed, let $g \in V$, $g = \sum_{k=1}^{n} \lambda_k \chi_{A_k}$, where $A_1, \ldots, A_n$ are pairwise disjoint sets . Then $|\lambda_k| \leq 1/r$ and so

$$p(\hat{m}(g)) = p(\sum_{k=1}^{n} \lambda_k m(A_k)) \leq \max_k |\lambda_k| \cdot p(m(A_k)) \leq 1.$$

This completes the proof.

Theorem 3.6 *ϕ is the finest of all Hausdorff locally convex topologies ρ on $S(\mathcal{R})$ such that, for $F = (S(\mathcal{R}), \rho)$, the map $\chi : \mathcal{R} \to F$ is in $M(\mathcal{R}, F)$. Analogous results hold for ϕ_σ and ϕ_τ.*

Proof : Let ρ be a Hausdorff locally convex topology on $S(\mathcal{R})$ such that $\chi : \mathcal{R} \to (S(\mathcal{R}), \rho)$ is a bounded finitely additive measure. By the definition of ϕ , the linear map

$$\hat{\chi} : (S(\mathcal{R}), \phi) \to (S(\mathcal{R}), \rho)$$

is continuous. Since $\hat{\chi}$ is the identity map, it follows that ϕ is finer that ρ. Thus the result holds for ϕ. Analogous are the proofs for ϕ_σ and ϕ_τ.

Corollary 3.7 *On $S(\mathcal{R})$ the topology ϕ coincides with the topology τ_u of uniform convergence.*

Proof : It follows from Theorems 3.5 and 3.6 since $\chi : \mathcal{R} \to (S(\mathcal{R}), \tau_u)$ is a bounded finitely-additive measure.

Let $\sigma = \sigma(M(\mathcal{R}), S(\mathcal{R}))$. For a σ-bounded subset H of $M(\mathcal{R})$, we denote by H_σ the set H equipped with the topology induced by σ. Let $C_b(H_\sigma)$ be the space of all bounded continuous $\mathbb{K}$-valued functions on H_σ endowed with the sup norm topology. For $A \in \mathcal{R}$, the function $m \mapsto m(A)$, $m \in H$, is σ-continuous. Also this function is bounded because H is σ-bounded. Hence we get a map

$$\mu = \mu_H : \mathcal{R} \to C_b(H_\sigma), \quad < \mu(A), m >= m(A).$$

Theorem 3.8 *For a subset H of $M(\mathcal{R})$, the following are equivalent :*

1. *H is ϕ-equicontinuous.*

2. *H is σ-bounded and the map $\mu = \mu_H : \mathcal{R} \to F = C_b(H_\sigma)$ is in $M(\mathcal{R}, F)$.*

Proof : (1) $\Rightarrow$ (2). Since H is ϕ-equicontinuous, it is σ-bounded. Clearly μ is finitely additive. We need to show that $\mu(\mathcal{R})$ is a norm bounded subset of $C_b(H_\sigma)$. Indeed, let V be a ϕ-neighborhood of zero in $S(\mathcal{R})$ such that $H \subset V^o$. Since $\chi : \mathcal{R} \to (S(\mathcal{R}), \phi)$ is a bounded measure, there exists a non-zero element λ of $\mathbb{K}$ such that $\chi_A \in \lambda V$ for all $A \in \mathcal{R}$. Thus, for $A \in \mathcal{R}$ and $m \in H$, we have that $|m(A)| \leq |\lambda|$ and hence $\|\mu(A)\| \leq |\lambda|$. Thus, $\sup_{A\in\mathcal{R}} \|\mu(A)\| \leq |\lambda|$, which proves that $\mu \in M(\mathcal{R}, F)$.
(2) $\Rightarrow$ (1). Since $\mu : \mathcal{R} \to F = C_b(H_\sigma)$ is a bounded finitely-additive measure, it follows that $\hat{\mu} : (S(\mathcal{R}), \phi) \to F$ is continuous. Thus, there exists a ϕ-neighborhood V of zero such that $\|\hat{\mu}(g)\| \leq 1$ for all $g \in V$. Then $H \subset V^o$ and the result follows.

Theorem 3.9 *For a subset H of $M_\sigma(\mathcal{R})$, the following are equivalent :*

1. *H is ϕ_σ-equicontinuous.*

2. *H is σ-bounded and the map $\mu = \mu_H : \mathcal{R} \to C_b(H_\sigma)$ is a σ-additive measure.*

3. *H is σ-bounded and uniformly σ-additive.*

4. $\sup_{m\in H} \|m\| < \infty$ *and H is uniformly σ-additive.*

Proof : (1) $\Rightarrow$ (2). Since $\phi_\sigma \subset \phi$, it follows that H is ϕ-equicontinuous and thus (by the preceding Theorem) $\mu : \mathcal{R} \to C_b(H_\sigma)$ is a bounded finitely-additive measure. We need to show that μ is σ-additive. So let (V_n) be a sequence of measurable sets which decreases to the empty set. Since H is ϕ_σ-equicontinuous, there exists a ϕ_σ-neighborhood V of zero in $S(\mathcal{R})$ such that $H \subset V^o$. Let $\lambda \neq 0$. As $\chi : \mathcal{R} \to (S(\mathcal{R}), \phi_\sigma)$ is a σ-additive measure, there exists n_o such that $\chi_{V_n} \in \lambda V$, for all $n \geq n_o$. Thus, for $n \geq n_o$ and $m \in H$, we have $|m(V_n)| \leq |\lambda|$ and thus $\|\mu(A_n\| \leq |\lambda|$, which proves that μ is σ-additive.
(2) $\Rightarrow$ (3). Let $V_n \downarrow \emptyset$. Since $\mu(V_n) \to 0$ in $C_b(H_\sigma)$, given $\epsilon > 0$, there exists n_o such that $\|\mu(V_n)\| \leq \epsilon$ for all $n \geq n_o$. Thus, for $n \geq n_o$, we have that $|m(V_n| \leq \epsilon$ for all $m \in H$, which proves that H is uniformly σ-additive.
(3) $\Rightarrow$ (2). It is trivial.
(2) $\Rightarrow$ (1). Since $\mu = \mu_H : \mathcal{R} \to C_b(H_\sigma)$ is a σ-additive measure, the map $\hat{\mu} : (S(\mathcal{R}), \phi_\sigma) \to F$ is continuous. Hence, there exists a ϕ_σ-neighborhood V of zero such that $\|\hat{\mu}(g)\| \leq 1$ for all $g \in V$. But then $H \subset V^o$.
(1) $\Rightarrow$ (4). Since ϕ_σ is coarser than the topology τ_u of uniform convergence, it follows that H is τ_u-equicontinuous and hence $\sup_{m\in H} \|m\| < \infty$. Also H is uniformly

σ-additive since (1) implies (3). This clearly completes the proof.

The proof of the next Theorem is analogous to the one of the preceding Theorem.

Theorem 3.10 *For a subset H of $M_\tau(\mathcal{R})$, the following are equivalent :*

1. *H is ϕ_τ-equicontinuous.*
2. *H is σ-bounded and the map $\mu = \mu_H : \mathcal{R} \to C_b(H_\sigma)$ is a τ-additive measure.*
3. *H is σ-bounded and uniformly τ-additive.*
4. *$\sup_{m \in H} \|m\| < \infty$ and H is uniformly τ-additive.*

Theorem 3.11 *ϕ_τ is the weakest of all locally convex topologies ρ on $S(\mathcal{R})$ such that, for each non-Archimedean Banach space E and each $m \in M_\tau(\mathcal{R}, E)$, the map $\hat{m} : (S(\mathcal{R}), \rho) \to E$ is continuous.*

Proof : Let τ_o be the weakest of all locally convex topologies ρ on $S(\mathcal{R})$ having the property mentioned in the Theorem. Clearly τ_o is coarser than ϕ_τ. On the other hand, let W be a polar ϕ_τ-neighborhood of zero and let H be the polar of W in $M_\tau(\mathcal{R})$. By the preceding Theorem,

$$\mu = \mu_H : \mathcal{R} :\to E = C_b(H_\sigma)$$

is a τ-additive measure. If V is the unit ball of E, then $(\hat{\mu})^{-1}(V)$ ia a τ_o-neighborhood of zero. Since $(\hat{\mu})^{-1}(V) \subset H^o = W$, the result clearly follows.

4 Integration

Throughout the rest of the paper we will assume that E is a complete Hausdorff locally convex space (unless it is stated otherwise) and $\mathcal{R}$ a separating algebra of subsets of a non-empty set X. Let $m \in M(\mathcal{R}, E)$ and $A \in \mathcal{R}$. Let $\mathcal{D}_A$ be the family of all $\alpha = \{A_1, A_2, \ldots, A_n; x_1, x_2, \ldots, x_n\}$, where $\{A_1, A_2, \ldots, A_n\}$ is a finite $\mathcal{R}$-partition of A and $x_i \in A_i$. We make $\mathcal{D}_A$ into a directed set by defining $\alpha_1 \geq \alpha_2$ iff the partition of A in α_1 is a refinement of the one in α_2. For $\alpha = \{A_1, A_2, \ldots, A_n; x_1, x_2, \ldots, x_n\} \in \mathcal{D}_A$ and $f \in \mathbb{K}^X$, we define

$$\omega_\alpha(f, m) = \sum_{k=1}^{n} f(x_k) m(A_k).$$

If the $\lim_\alpha \omega_\alpha(f, m)$ exists in E, we will say that f is m-integrable over A and denote this limit by $\int_A f\, dm$. For $A = X$, we write simply $\int f\, dm$. It is easy to see that, if f is m-integrable over X, then it is m-integrable over every measurable subset A and $\int_A f\, dm = \int f \chi_A\, dm$. If f is bounded on A, then $p\left(\int_A f\, dm\right) \leq \|f\|_A \cdot m_p(A)$ for every $p \in cs(E)$.
Using an argument analogous to the one used in [6], Theorem 2.1 for scalar-valued measures, we get the following

Theorem 4.1 *If $m \in M(\mathcal{R}, E)$, then an $f \in \mathbb{K}^X$ is m-integrable iff, for each $p \in cs(E)$ and each $\epsilon > 0$, there exists an $\mathcal{R}$-partition $\{A_1, A_2, \ldots, A_n\}$ of X such that $|f(x) - f(y)| \cdot m_p(A_i) \leq \epsilon$, for all i, if the x, y are in A_i. Moreover, in this case we have that*

$$p\left(\int f\, dm - \sum_{i=1}^{n} f(x_i) m(A_i)\right) \leq \epsilon.$$

Theorem 4.2 *Let $m \in M(\mathcal{R}, E)$ and let $f \in \mathbb{K}^X$ be m-integrable. Then :*

1. *f is continuous at every x in the set*

$$D = \bigcup_{p \in cs(E)} \{x : N_{m,p}(x) \neq 0\}.$$

2. *For each $p \in cs(E)$, there exists a measurable set A, with $m_p(A^c) = 0$, such that f is bounded on A.*

Proof : (1). Suppose that $N_{m,p}(x) = d > 0$ and let $\epsilon > 0$. There exists an $\mathcal{R}$-partition $\{A_1, A_2, \ldots, A_n\}$ of X such that $|f(x) - f(y)| \cdot m_p(A_i) \leq d\epsilon$, if $x, y \in A_i$. If $x \in A_i$, then $|f(y) - f(x)| \leq \epsilon$ for all $y \in A_i$.
(2). Let $\{A_1, A_2, \ldots, A_n\}$ be an $\mathcal{R}$-partition of X such that $|f(x) - f(y)| \cdot m_p(A_i) \leq 1$, if $x, y \in A_i$. Let

$$A = \bigcup \{A_i : m_p(A_i) > 0\}.$$

It follows easily that f is bounded on A and that $m_p(A^c) = 0$.

Theorem 4.3 *Let $m \in M(\mathcal{R}, E)$. If $f, g \in \mathbb{K}^X$ are m-integrable, then $h = fg$ is also m-integrable.*

Proof : Let $p \in cs(E)$ and $\epsilon > 0$. There are measurable sets A, B such that $m_p(A^c) = m_p(B^c) = 0$ and f, g are bounded on A, B, respectively. Let $D = A \cap B$. Then $m_p(D^c) = 0$ and there exists a $d > 0$ such that $\|f\|_D$, $\|g\|_D \leq d$. Now there exists an $\mathcal{R}$-partition $\{A_1, A_2, \ldots, A_n\}$ of X, which is a refinement of $\{D, D^c\}$, such that

$$|f(x) - f(y)| \cdot m_p(A_i) < \epsilon/d \quad \text{and} \quad |g(x) - g(y)| \cdot m_p(A_i) < \epsilon/d$$

if $x, y \in A_i$. Let now $x, y \in A_i$. If $A_i \subset D^c$, then $|h(x) - h(y)| \cdot m_p(A_i) = 0$. For $A_i \subset D$, we have that

$$\begin{aligned} |h(x) - h(y)| &= |[f(x) - f(y)]g(x) + f(y)[g(x) - g(y)]| \\ &\leq \max\{d \cdot |f(x) - f(y)|, d \cdot |g(x) - g(y)|\} \end{aligned}$$

and so $|h(x) - h(y)| \cdot m_p(A_i < \epsilon$. This completes the proof in view of Theorem 4.1.

Let now $m \in M(\mathcal{R}, E)$ and let $g \in \mathbb{K}^X$ be m-integrable. Define

$$m_g : \mathcal{R} \to E, \quad m_g(A) = \int_A g\, dm.$$

Clearly m_g is finitely-additive. Also, m_g is bounded. In fact, let $p \in cs(E)$. There exists a measurable set B such that $m_p(B^c) = 0$ and g is bounded on B. Let $d = \|g\|_B$. Let $A \in \mathcal{R}$, $W_1 = A \cap B$, $W_2 = A \cap B^c$. Since g is m-integrable, there exists an $\mathcal{R}$-partition $\{V_1, V_2, \ldots, V_n\}$ of A, which is a refinement of $\{W_1, W_2\}$ such that $|g(x) - g(y)| \cdot m_p(V_i) < 1$ if $x, y \in V_i$. Let $x_i \in V_i$. Then

$$p\left(\int_A g\,dm - \sum_{k=1}^{n} g(x_i)m(V_i)\right) < 1.$$

If $V_i \subset W_1$, then $p\left(g(x_i)m(V_i)\right) \leq d \cdot m_p(X)$, while for $V_i \subset W_2$ we have that $p\left(g(x_i)m(V_i)\right) = 0$. Thus

$$p\left(\int_A g\,dm\right) \leq \max\{1,\, d \cdot m_p(X)\}.$$

This proves that m_g is bounded and hence $m_g \in M(\mathcal{R}, E)$.

Theorem 4.4 *Let $m \in M(\mathcal{R}, E)$ and let $g \in \mathbb{K}^X$ be m-integrable. If $f \in \mathbb{K}^X$ is m-integrable, then f is m_g-integrable and $\int f\,dm_g = \int fg\,dm$.*

Proof : Let $p \in cs(E)$. There exists a measurable set D, with $m_p(D^c) = 0$, such that f, g are bounded on D. Let $d > \max\{\|f\|_D, \|g\|_D\}$. If V is a measurable set contained in D^c, then $p(m_g(V)) = 0$. This follows from the fact that, for $A \subset V$ we have that $p(g(x)m(A)) = 0$. Let now $\epsilon > 0$ be given. There exists an $\mathcal{R}$-partition $\{V_1, V_2, \ldots, V_N\}$ of X , which is a refinement of $\{D, D^c\}$, such that

$$|f(x) - f(y)| \cdot m_p(V_i) < \epsilon/d, \quad \textit{and} \quad |g(x) - g(y)| \cdot m_p(V_i) < \epsilon/d$$

if $x, y \in V_i$. We may assume that $\bigcup_{i=1}^n V_i = D$. For $A \in \mathcal{R}$, $A \subset V_i \subset D$, we have

$$p\left(\int_A g\,dm\right) \leq \|g\|_A \cdot m_p(A) \leq d \cdot m_p(V_i),$$

and hence $(m_g)_p(V_i) \leq d \cdot m_p(V_i)$. Thus, for $x, y \in V_i \subset D$, we have

$$|f(x) - f(y)| \cdot (m_g)_p(V_i) \leq d \cdot |f(x) - f(y)| \cdot m_p(V_i) \leq \epsilon.$$

The same inequality holds when $V_i \subset D^c$. This proves that f is m_g-integrable. If $x, y \in V_i \subset D$, then

$$p\left(\int f\,dm_g - \sum_{k=1}^{n} f(x_k)m_g(V_k)\right) \leq \epsilon.$$

Since, for $x, y \in V_k \subset D$, we have $|g(x) - g(y)| \cdot m_p(V_k) \leq \epsilon/d$, it follows that

$$p\left(m_g(V_k) - g(x_k)m(V_k)\right) \leq \epsilon/d.$$

For $x, y \in V_k \subset D$, we have

$$|f(x)g(x) - f(y)g(y)| \cdot m_p(V_k) \leq m_p(V_k) \cdot \max\{|g(x)| \cdot |f(x) - f(y)|, |f(y)| \cdot |g(x) - g(y)|\} \leq \epsilon.$$

Since $m_p(V_k) = 0$ if $V_k \subset D^c$, we get that

$$p\left(\int gf\,dm - \sum_{k=1}^{n} g(x_k)f(x_k)m(V_k)\right) \leq \epsilon.$$

Also, for $1 \leq k \leq n$, we have $p\left(f(x_k)g(x_k)m(V_k) - f(x_k)m_g(V_k)\right) \leq \epsilon$. It follows that

$$p\left(\int gf\,dm - \int f\,dm_g\right) \leq \epsilon.$$

This, being true for all $\epsilon > 0$, and the fact that E is Hausdorff, imply that

$$\int gf\,dm = \int f\,dm_g,$$

which completes the proof.

Theorem 4.5 *Let $m \in M(\mathcal{R}, E)$, $p \in cs(E)$ and $x \in X$. If $g \in \mathbb{K}^X$ is m-integrable, then*

$$N_{m_g,p}(x) = |g(x)| \cdot N_{m,p}(x).$$

Proof : Let $\epsilon > 0$. There exists an $\mathcal{R}$-partition $\{V_1, V_2, \cdots, V_n\}$ of X such that $|g(y) - g(z)| \cdot m_p(V_i) \leq \epsilon$ if $y, z \in V_i$.

Claim I : If V is a measurable subset of V_i containing x, then, for each $A \subset V$, we have

$$p(m_g(A)) \leq \max\{\epsilon, |g(x)| \cdot m_p(V)\} = \theta.$$

Indeed, if $x \in A$, then for each $y \in A$ we have that $|g(x) - g(y)| \cdot m_p(A) \leq \epsilon$, which implies that $p(m_g(A) - g(x)m(A)) \leq \epsilon$ and so

$$p(m_g(A)) \leq \max\{\epsilon, |g(x)| \cdot p(m(A))\} \leq \theta.$$

In case $x \in V \backslash A$, we get in the same way that $p(m_g(V \backslash A)) \leq \theta$. Also $p(m_g(V)) \leq \theta$, since $x \in V$. Thus

$$p(m_g(A)) = p(m_g(V) - m_g(V \setminus A)) \leq \theta,$$

and the claim follows.

Claim II. If W is a measurable subset of V_i containing x, then for each measurable set $A \subset W$, we have that

$$|g(x)| \cdot p(m(A)) \leq \max\{\epsilon, (m_g)_p(W)\} = d.$$

Indeed, if $x \in A \subset W$, then $p(m_g(A) - g(x)m(A)) \leq \epsilon$ and so

$$|g(x)| \cdot p(m(A)) \leq \max\{\epsilon, p(m_g(A))\} \leq d.$$

If $x \in W \setminus A$, then $|g(x)| \cdot p(m(W \setminus A)) \leq d$. Also $g(x)| \cdot p(m(W)) \leq d$, and so again $|g(x)| \cdot p(m(A)) \leq d$, which proves the claim.

Now there are measurable subsets V, W of V_i containing x such that

$$m_p(V) < N_{m,p}(x) + \epsilon, \quad \textit{and} \quad (m_g)_p(W) < \epsilon + N_{m_g,p}(x).$$

By claim I, we have

$$\begin{aligned} N_{m_g,p}(x) \leq (m_g)_p(V) &\leq \max\{\epsilon, |g(x)| \cdot m_p(V)\} \\ &\leq \max\{\epsilon, |g(x)|[\epsilon + N_{m,p}(x)]\}. \end{aligned}$$

Taking $\epsilon \to 0$, we get that

$$N_{m_g,p}(x) \leq |g(x)| \cdot N_{m,p}(x).$$

Also

$$|g(x)| \cdot N_{m,p}(x) \leq |g(x)| \cdot m_p(W) \leq \max\{\epsilon, (m_g)_p(W)\} < \epsilon + N_{m_g,p}(x).$$

Taking $\epsilon \to 0$, we get that

$$|g(x)| \cdot N_{m,p}(x) \leq N_{m_g,p}(x),$$

which completes the proof.

Theorem 4.6 *Let $m \in M(\mathcal{R}, E)$ and let $g \in \mathbb{K}^X$ be m-integrable. If m is τ-additive (resp. σ-additive), then m_g is τ-additive (resp. σ-additive).*

Proof : Assume that m is τ-additive and let $V_\delta \downarrow \emptyset$ and $p \in cs(E)$ There exists an $A \in \mathcal{R}$ such that $m_p(A^c) = 0$ and $\|f\|_A = d < \infty$. Given $\epsilon > 0$, there exists a δ_o such that $m_p(V_\delta) < \epsilon/d$ if $\delta \geq \delta_o$. For a measurable set V disjoint from A, we have $p(m_g(V)) = 0$. Thus, for $\delta \geq \delta_o$, we have

$$p(m_g(V_\delta)) = p(m_g(V_\delta \cap A)) \leq \|g\|_{V_\delta \cap A} \cdot m_p(V_\delta \cap A) \leq d \cdot m_p(V_\delta) < \epsilon.$$

This proves that m_g is τ-additive. The proof for the σ-additive case is analogous.

The proof of the next Theorem is analogous to the one given in [8], Theorem 2.16, for scalar-valued measures.

Theorem 4.7 *Let $m \in M(\mathcal{R}, E)$. For a subset Z of X, the following are equivalent:*

1. *χ_Z is m-integrable.*

2. *For each $p \in cs(E)$ and each $\epsilon > 0$, there are measurable sets V, W such that $V \subset Z \subset W$ and $m_p(W \setminus V) < \epsilon$.*

For $m \in M(\mathcal{R}, E)$, let $\mathbb{R}_m$ be the family of all $A \subset X$ such that χ_A is m-integrable. Using the preceding Theorem, we show easily that $\mathbb{R}_m$ is a separating algebra of subsets of X which contains $\mathcal{R}$. Define

$$\overline{m} : \mathbb{R}_m \to E, \quad \overline{m}(A) = \int \chi_A \, dm.$$

The proofs of the next two Theorems are analogous to the corresponding ones for scalar valued measures (see [8], Lemma 2.18, Theorems 2.22, 2.23, 2.24, 2.26 and Corollary 2.25).

Theorem 4.8 *1. For $A \in \mathcal{R}$ and $p \in cs(E)$, we have $m_p(A) = \overline{m}_p(A)$.*

2. $\overline{m}$ is σ-additive iff m is σ-additive.

3. $\overline{m}$ is τ-additive iff m is τ-additive.

4. $N_{m,p} = N_{\overline{m},p}$.

5. $\mathbb{R}_m = \mathbb{R}_{\overline{m}}$.

Theorem 4.9 *1. If $f \in \mathbb{K}^X$ is m-integrable, then f is also $\overline{m}$-integrable and $\int f\, dm = \int f\, d\overline{m}$.*

2. If $f \in \mathbb{K}^X$ is $\overline{m}$-integrable and bounded, then f is also m-integrable.

Lemma 4.10 *If $m \in M_\tau(\mathcal{R}, E)$, then every $\tau_{\mathcal{R}}$-clopen set A is in $\mathbb{R}_m$.*

Proof : Let $p \in cs(E)$ and $\epsilon > 0$. Consider the collection $\mathcal{F}$ of all $\mathcal{R}$-measurable sets of the form $W \setminus V$, where $V, W \in \mathcal{R}$ and $V \subset A \subset W$. Then $\mathcal{F} \downarrow \emptyset$. As m is τ-additive, there exists an $W \setminus V \in \mathcal{F}$ such that $m_p(W \setminus V) < \epsilon$, which proves that $A \in \mathbb{R}_m$.

Theorem 4.11 *Let $m \in M_\tau(\mathcal{R}, E)$ and $f \in \mathbb{K}^X$. If f is bounded and $\tau_{\mathcal{R}}$-continuous, then f is m-integrable (and hence $\overline{m}$-integrable).*

Proof : Without loss of generality, we may assume that $\|f\| \leq 1$. Let $p \in cs(E)$ and $\epsilon > 0$. The set $Y = \{x : N_{m,p}(x) \geq \epsilon\}$ is $\tau_{\mathcal{R}}$-compact. Choose $0 < \epsilon_1 < \epsilon$ such that $\epsilon_1 . m_p(X) < \epsilon$. There are $x_1, x_2, \cdots, x_n \in Y$ such that the sets

$$A_k = \{x : p(f(x) - f(x_k)) \leq \epsilon_1\}, \quad k = 1, \cdots, n.$$

are pairwise disjoint and cover Y. Each A_k is $\tau_{\mathcal{R}}$-clopen and hence it is a member of $\mathbb{R}_m$. Let $V_k, W_k \in \mathcal{R}$ be such that $V_k \subset A_k \subset W_k$ and $m_p(W_k \setminus V_k) < \epsilon$. Let $V_{n+1} = (\bigcup_{k=1}^n V_k)^c$. Then V_{n+1} is disjoint from Y. Indeed, if $x \in Y \cap V_{n+1}$, then $x \in W_k$, for some k, and so $N_{m,p}(x) \leq m_p(W_k \setminus V_k) < \epsilon$, a contradiction. As m is τ-additive, we have that $m_p(V_{n+1}) = \sup_{x \in V_{n+1}} N_{m,p}(x) \leq \epsilon$. If now $x, y \in V_i$, $i \leq n$, then

$$|f(x) - f(y)| \cdot m_p(V_i) \leq \epsilon_1 . \cdot m_p(X) < \epsilon.$$

Also, if $x, y \in V_{n+1}$, then $|f(x) - f(y)| \cdot m_p(V_{n+1}) \leq \epsilon$. This proves that f is m-integrable.

Theorem 4.12 *Let $m \in M_\tau(\mathcal{R}, E)$. For a subset A of X, the following are equivalent :*

1. $A \in \mathbb{R}_m$.

2. A is $\tau_{\mathbb{R}_m}$-clopen.

Proof : Clearly (1) $\Rightarrow$ (2). On the other hand let A be $\tau_{\mathbb{R}_m}$-clopen. Since $\overline{m}$ is τ-additive, it follows (by Theorem 4.11) that χ_A is $\overline{m}$-integrable and hence χ_A is m-integrable (by Theorem 4.9), which means that $A \in \mathbb{R}_m$.

Theorem 4.13 *Let $m \in M_\tau(\mathcal{R}, E)$ and consider on X the topology $\tau_{\mathcal{R}}$. Then the map*

$$u_m : C_b(X) \to E, \quad u_m(f) = \int f\, dm = \int f\, d\overline{m}$$

is β-continuous. Also, every β-continuous linear map $u : C_b(X) \to E$ is of the form $u = u_m$ for some $m \in M_\tau(\mathcal{R}, E)$.

Proof : Let $p \in cs(E)$ and $G \in \Omega$. We need to show that the set

$$V = \{f \in C_b(X) : p(u_m(f)) \leq 1\}$$

is a β_G-neighborhood of zero. Indeed, let $r > 0$. There exists a decreasing net (V_δ) of $\tau_{\mathcal{R}}$-clopen sets with $\bigcap_\delta \overline{V_\delta}^{\beta_o X} = G$. Since $V_\delta \in \mathbb{R}_m$ and $\overline{m}$ is τ-additive, there exists a δ such that $\overline{m}_p(V_\delta) < 1/r$. Now

$$V_1 = \{f \in C_b(X) : \|f\| \leq r, \quad \|f\|_{V_\delta^c} \leq 1/\|m\|_p\} \subset V.$$

In fact, let $f \in V_1$ and set $h = f\chi_{V_\delta}$, $g = f\chi_{V_\delta^c}$. Then

$$p\left(\int h\, dm\right) = p\left(\int h\, d\overline{m}\right) \leq \|h\| \cdot \overline{m}_p(V_\delta) \leq 1$$

and

$$p\left(\int g\, dm\right) = p\left(\int g\, d\overline{m}\right) \leq \|f\|_{V_\delta^c} \cdot \overline{m}_p(X) \leq 1.$$

Thus $p(\int f\, dm) \leq 1$, which shows that $V_1 \subset V$. Since the closure of V_δ^c in $\beta_o X$ is disjoint from G, this proves that V is a β_G-neighborhood of zero. This, being true for every $G \in \Omega$, implies that V is a β-neighborhood of zero and so u_m is β-continuous. Conversely let $u : (C_b(X), \beta) \to E$ be linear and continuous. Since β is coarser than the topology of uniform convergence, it follows that, for each $p \in cs(E)$, there exists a non-zero $\lambda \in \mathbb{K}$ such that

$$\{f \in C_b(X) : \|f\| \leq |\lambda|\} \subset \{f : p(u(f)) \leq 1\}.$$

Let $K(X)$ be the algebra of al $\tau_{\mathcal{R}}$-clopen subsets of X. Define

$$\mu : K(X) \to E, \quad \mu(A) = u(\chi_A).$$

Clearly μ is finitely-additive. Also, since $|\lambda\chi_A| \leq |\lambda|$, it follows that $p(\mu(A)) \leq |\lambda|^{-1}$, and so μ is bounded. If (V_δ) is a net of clopen sets which decreases to the empty set, then $\chi_{V_\delta} \to 0$ with respect to the topology β and so $\mu(V_\delta) \to 0$. Thus $\mu \in M_\tau(K(X), E)$. The restriction $m = \mu|_{\mathcal{R}}$ is in $M_\tau(\mathcal{R}, E)$. The subspace F of $C_b(X)$ spanned by the functions χ_A, $A \in K(X)$, is β-dense in $C_b(X)$. Since u and u_m are both β-continuous and they coincide in F, it follows that $u = u_m$ on $C_b(X)$. This completes the proof.

Theorem 4.14 *Let X be a zero-dimensional Hausdorff topological space and E a Hausdorff locally convex space. Then a linear map $u : C_b(X) \to E$ is β-continuous iff it is β_o-continuous.*

Proof : Let $\hat{E}$ be the completion of E and let $K(X)$ be the algebra of all clopen subsets of X. Suppose that u is β-continuous. Then $u : (C_b(X), \beta) \to \hat{E}$ is continuous. In view of the preceding Theorem, there exists an $m \in M_\tau(K(X), \hat{E})$ such that $u(f) = \int f\,dm$ for all $f \in C_b(X)$. Let $p \in cs(E)$ and

$$V = \{f : p(u(f)) \leq 1\}.$$

We need to show that V is a β_o-neighborhood of zero. By [4], Theorem 2.8, it suffices to show that, for each $r > 0$, there exists a compact subset Y of X and $\epsilon > 0$ such that

$$V_1 = \{f \in C_b(X) : \|f\| \leq r,\ \|f\|_Y \leq \epsilon\} \subset V.$$

Choose $\epsilon > 0$ such that $\epsilon \cdot m_p(X) \leq 1$ and $r \cdot \epsilon \leq 1$. The set $X_{p,\epsilon} = \{x : N_{m,p}(x) \geq \epsilon\}$ is compact. In the definition of V_1 take as Y the set $X_{p,\epsilon}$. Let $f \in V_1$ and $A = \{x : |f(x)| \leq \epsilon\}$. Then $m_p(A^c) = \sup_{x \in A^c} N_{m,p}(x) \leq \epsilon$. Now

$$p\left(\int_A f\,dm\right) \leq \epsilon \cdot m_p(X) \leq 1, \quad \textit{and} \quad p\left(\int_{A^c} f\,dm\right) \leq r \cdot m_p(A^c) \leq 1.$$

Thus $V_1 \subset V$ and the result follows.

Theorem 4.15 *Let $\mathcal{R}$ be a separating algebra of subsets of a set X and consider on X the topology $\tau_{\mathcal{R}}$. Then ϕ_τ coincides with the topology induced on $S(\mathcal{R})$ by β_o and by the topology induced by β.*

Proof : If (V_δ) is a net of measurable subsets of X which decreases to the empty set, then $\chi_{V_\delta} \downarrow 0$ and so $\chi_{V_\delta} \xrightarrow{\beta} 0$. Thus

$$\chi : \mathcal{R} \to (S(\mathcal{R}), \beta)$$

is a τ-additive measure. In view of Theorem 3.6 , it follows that ϕ_τ is finer than the topology induced on $S(\mathcal{R})$ by β. On the other hand, let E be a Hausdorff locally convex space and let $\hat{E}$ be its completion. If $m \in M_\tau(\mathcal{R}, E)$, then $m \in M_\tau(\mathcal{R}, \hat{E})$. The map

$$u : C_b(X) \to \hat{E}, \quad u(f) = \int f\,dm,$$

is β_o-continuous. Since $\hat{m} = u|_{S(\mathcal{R})}$, it follows that $\hat{m} : (S(\mathcal{R}), \beta_o) \to \hat{E}$ is continuous and hence $\hat{m} : (S(\mathcal{R}), \beta_o) \to E$ is continuous. This implies that ϕ_τ is coarser than the topology induced on $S(\mathcal{R})$ by β_o and the result follows.

Corollary 4.16 *The topology ϕ_τ is polar and locally solid.*

Lemma 4.17 *Let Z be a vector space over $\mathbb{K}$, D a subspace of Z and τ_1, τ_2 Hausdorff locally convex topologies on Z which induce the same topology on D and for both of which D is dense in Z. If τ_2 is finer than τ_1, then τ_1 and τ_2 coincide on Z.*

Proof: Let $G = (Z, \tau_2)$ and let $\hat{G}$ be its completion. The identity map $T : (Z, \tau_2) \to \hat{G}$ is clearly continuous. Let $S = T|_D$. Since τ_1 and τ_2 induce the same topology on D, it follows that $S : (D, \tau_1) \to \hat{G}$ is continuous. As D is τ_1-dense in Z, there exists a unique continuous extension $\hat{S} : (Z, \tau_1) \to \hat{G}$. Now $\hat{S} : (Z, \tau_2) \to \hat{G}$ is continuous. Since $\hat{S} = T$ on D and D is τ_2-dense in Z, it follows that $\hat{S} = T$ on Z. Thus

$$T = \hat{S} : (Z, \tau_1) \to \hat{G}$$

is continuous, which clearly implies that τ_1 is finer than τ_2 and the Lemma follows.

Theorem 4.18 *For any zero-dimensional Hausdorff topological space X, the topologies β and β_o coincide on $C_b(X)$.*

Proof : Let $K(X)$ be the algebra of all clopen subsets of X. Since $S(K(X))$ is β-dense in $C_b(X)$, the result follows from Theorem 4.15 and the preceding Lemma.

Theorem 4.19 *Let Δ be the family of all pairs (m, p) for which there exists a Hausdorff locally convex space E such that $p \in cs(E)$ and $m \in M_\tau(\mathcal{R}, E)$. To each $\delta = (m, p) \in \Delta$ corresponds the non-Archimedean seminorm $\|\cdot\|_{N_{m,p}}$ on $S(\mathcal{R})$. Then ϕ_τ coincides with the locally convex topology ρ generated by these seminorms.*

Proof : Let E be a Hausdorff locally convex space, $m \in M_\tau(\mathcal{R}, E)$ and $p \in cs(E)$. If $g = \sum_{k=1}^n \alpha_k \chi_{A_k} \in S(\mathcal{R})$, then

$$p(\hat{m}(g)) = p\left(\sum_{k=1}^n \alpha_k m(A_k)\right) \leq \max_k |\alpha_k| \cdot p(m(A_k)) \leq \|g\|_{N_{m,p}}.$$

Thus $\hat{m} : S(\mathcal{R}), \rho) \to E$ is continuous and so ϕ_τ is coarser than ρ. On the other hand, let $(m, p) \in \Delta$ and

$$V = \{g \in S(\mathcal{R}) : p(\hat{m}(g)) \leq 1\}.$$

Since ϕ_τ is locally solid, there exists a solid ϕ_τ-neighborhood V_1 of zero contained in V. Now $V_1 \subset \{g : \|g\|_{N_{m,p}} \leq 1\}$. In fact, assume that, for some $g = \sum_{k=1}^n \alpha_k \chi_{A_k} \in V_1$, we have that $\|g\|_{N_{m,p}} > 1$. There exists an x in some A_k such that

$$|g(x)| \cdot N_{m,p}(x) = |\alpha_k| \cdot N_{m,p}(x) > 1.$$

There is a measurable set A contained in A_k such that $|\alpha_k| \cdot p(m(A)) > 1$. If $h = \alpha_k \chi_A$, then $|h| \leq |g|$ and so $h \in V_1$, which is a contradiction since $p(\hat{m}(h)) > 1$. This contradiction shows that

$$V_1 \subset \{g : \|g\|_{N_{m,p}} \leq 1\}.$$

Thus ρ is coarser than ϕ_τ and the result follows.

5 (VR)-Integrals

Throughout this section, $\mathcal{R}$ will be a separating algebra of subsets of a set X, E a complete Hausdorff locally convex space and $m \in M_\tau(\mathcal{R}, E)$. For $p \in cs(E)$, and $f \in \mathbb{K}^X$, let

$$\|f\|_{N_{m,p}} = \sup_{x \in X} |f(x)| \cdot N_{m,p}(x).$$

Let G_m be the space of all $f \in \mathbb{K}^X$ for which $\|f\|_{N_{m,p}} < \infty$, for each $p \in cs(E)$. Each $\|.\|_{N_{m,p}}$ is a non-Archimedean seminorm on G_m. We will consider on G_m the locally convex topology generated by these seminorms.

Lemma 5.1 *If $g = \sum_{k=1}^n \alpha_k \chi_{A_k} \in S(\mathcal{R})$, then*

$$p\left(\sum_{k=1}^n \alpha_k m(A_k)\right) \leq \|g\|_{N_{m,p}}.$$

Proof : We first observe that

$$\|g\|_{N_{m,p}} \leq \|g\| \cdot m_p(X) < \infty.$$

If $g = \alpha \cdot \chi_A$, where $\alpha \in \mathbb{K}$ and $A \in \mathcal{R}$, then

$$\begin{aligned} p(\alpha \cdot m(A)) \leq |\alpha| \cdot m_p(A) &= |\alpha| \cdot \sup_{x \in A} N_{m,p}(x) \\ &= \sup_{x \in X} |g(x)| \cdot N_{m,p}(x) = \|g\|_{N_{m,p}}. \end{aligned}$$

In the general case, we may assume that the sets A_k, $k = 1, \ldots, n$, are pairwise disjoint. Then

$$p\left(\sum_{k=1}^n \alpha_k \cdot m(A_k)\right) \leq \max_k |\alpha_k| \cdot m_p(A_k) = \max_k \sup_{x \in A_k} |g(x)| \cdot N_{m,p}(x) = \|g\|_{N_{m,p}}.$$

Lemma 5.2 *If we consider on $S(\mathcal{R})$ the topology induced by the topology of G_m, then*

$$\omega : S(\mathcal{R}) \to E, \quad \omega(g) = \int g\, dm$$

is a continuous linear map.

Proof : It follows from the preceding Lemma.

Let now $\overline{S(\mathcal{R})}$ be the closure of $S(\mathcal{R})$ in G_m and let

$$\overline{\omega} : \overline{S(\mathcal{R})} \to E$$

be the unique continuous extension of ω.

Definition 5.3 *A function $f \in \mathbb{K}^X$ is said to be (VR)-integrable with respect to m if it belongs to $\overline{S(\mathcal{R})}$. In this case, $\overline{\omega}(f)$ is called the (VR)-integral of f, with respect to m, and will be denoted by $(VR)\int f\, dm$. We will denote by $L(m)$ the space $\overline{S(\mathcal{R})}$.*

Theorem 5.4 *If f is (VR)-integrable, then, for each $p \in cs(E)$, we have*

$$p\left((VR)\int f\,dm\right) \leq \|f\|_{N_{m,p}}.$$

Proof : There exists a ne (g_δ) in $S(\mathcal{R})$ such that $g_\delta \to f$ in $\overline{S(\mathcal{R})}$. Then

$$(VR)\int f\,dm = \lim_\delta \int g_\delta\,dm, \quad and \quad \|g_\delta\|_{N_{m,p}} \to \|f\|_{N_{m,p}}.$$

Since

$$p\left(\int g_\delta\,dm\right) \leq \|g_\delta\|_{N_{m,p}},$$

the result follows.

Theorem 5.5 *The space G_m is complete and hence $L(m)$ is also complete.*

Proof : Let (f_δ) be a Cauchy net in G_m and let

$$A = \bigcup_{p\in cs(E)} \{x : N_{m,p}(x) > 0\}.$$

Let $x \in A$ and choose $p \in cs(E)$ such that $N_{m,p}(x) = d > 0$. Given $\epsilon > 0$, there exists a δ_o such that $\|f_\delta - f_{\delta'}\|_{N_{m,p}} < d\epsilon$ if $\delta, \delta' \geq \delta_o$. Now, for $\delta, \delta' \geq \delta_o$, we have $|f_\delta(x) - f_{\delta'}(x)| < \epsilon$. This proves that the net $(f_\delta(x))$ is Cauchy in $\mathbb{K}$. Define

$$f(x) = \lim_\delta f_\delta(x), \quad if \quad x \in A$$

and $f(x)$ arbitrarily if $x \notin A$. We will show that $f \in G_m$ and that $f_\delta \to f$. Indeed, given $p \in cs(E)$ and $\epsilon > 0$, there exists δ_o such that

$$|f_\delta(x) - f_{\delta'}(x)| \cdot N_{m,p}(x) < \epsilon$$

for all x and all $\delta, \delta' \geq \delta_o$. Let now $\delta \geq \delta_o$ be fixed. If $x \in A$, then taking the limits on δ', we get that $|f_\delta(x) - f(x)| \cdot N_{m,p}(x) \leq \epsilon$. The same inequality also holds when $x \notin A$. Thus, for all $\delta \geq \delta_o$, we have

$$\sup_{x\in X} |f_\delta(x) - f(x)| \cdot N_{m,p}(x) \leq \epsilon.$$

It follows from this that , for all $x \in X$, we have

$$|f(x)| \cdot N_{m,p}(x) \leq \max\{\epsilon, \|f_{\delta_o}\|_{N_{m,p}}\}$$

which proves that $f \in G_m$. Also, $\|f - f_\delta\|_{N_{m,p}} \leq \epsilon$ for $\delta \geq \delta_o$. Hence $f_\delta \to f$ and the proof is complete.

Theorem 5.6 *For a subset A of X, the following are equivalent:*

1. *χ_A is (VR)-integrable.*

2. *For each* $p \in cs(E)$ *and each* $\epsilon > 0$, *there exists* $V \in \mathcal{R}$ *such that* $N_{m,p} < \epsilon$ *on* $A \triangle V$.

3. *For each* $p \in cs(E)$ *and each* $\epsilon > 0$, *there exists* $V \in \mathcal{R}$ *such that*

$$V \cap X_{p,\epsilon} = A \cap X_{p,\epsilon}.$$

Proof : (1) $\Leftrightarrow$ (2). The proof is analogous to the one given in [13], Lemma 7.3 for scalar valued measures.

(2) $\Leftrightarrow$ (3). It follows from the fact that, for $V \in \mathcal{R}$, $V \cap X_{p,\epsilon} = A \cap X_{p,\epsilon}$ iff $N_{m,p} < \epsilon$ on $A \triangle V$.

Let now $\tilde{R}_m$ be the family of all subsets A of X for which χ_A is (VR)-integrable with respect to m. It is easy to see that $\tilde{R}_m$ is a separating algebra of subsets of X which contains $\mathcal{R}$. Let $\tau_{\tilde{R}_m}$ be the zero dimensional topology having $\tilde{R}_m$ as a basis. In view of Theorem 2.5, for all $p \in cs(E)$ and all $\epsilon > 0$, the set $X_{p,\epsilon} = \{x : N_{m,p}(x) \geq \epsilon\}$ is $\tau_{\mathcal{R}}$-compact. Since $A \in \tilde{R}_m$ iff , for all $p \in cs(E)$ and all $\epsilon > 0$, there exists $V \in \mathcal{R}$ such that $V \cap X_{p,\epsilon} = A \cap X_{p,\epsilon}$, it follows that $X_{p,\epsilon}$ is $\tau_{\tilde{R}_m}$-compact. Also, since $\tau_{\mathcal{R}}$ is Hausdorff, $\tau_{\mathcal{R}}$ and $\tau_{\tilde{R}_m}$ induce the same topology on $X_{p,\epsilon}$. Now we define

$$\tilde{m} : \tilde{R}_m \to E, \quad \tilde{m}(A) = (VR) \int_A \chi_A \, dm.$$

Clearly $\tilde{m}$ is finitely-additive. Also $\tilde{m}$ is bounded since, for each $p \in cs(E)$, we have

$$p(\tilde{m}(A)) \leq \sup_{x \in A} N_{m,p}(x) \leq m_p(X).$$

Thus $\tilde{m} \in M(\tilde{R}_m, E)$.

Lemma 5.7 *If* $V \in \mathcal{R}$, *then* $m_p(V) = \tilde{m}_p(V)$.

Proof : It is clear that $m_p(V) \leq \tilde{m}_p(V)$. Suppose that $\tilde{m}_p(V) > \theta > 0$. There exists $A \in \tilde{R}_m$, $A \subset V$, $p(\tilde{m}(A)) > \theta$. Since $p(\tilde{m}(A)) \leq \sup_{x \in A} N_{m,p}(x)$, there exists $x \in A$ such that $N_{m,p}(x) > \theta$ and so $m_p(V) \geq N_{m,p}(x) > \theta$. This proves that $m_p(V) \geq \tilde{m}_p(V)$ and the Lemma follows.

Lemma 5.8 $N_{m,p} = N_{\tilde{m},p}$.

Proof : Since $m_p(V) = \tilde{m}_p(V)$ for $V \in \mathcal{R}$, it follows that $N_{m,p} \geq N_{\tilde{m},p}$. Assume that there exists an x such that $N_{m,p}(x) > \theta > N_{\tilde{m},p}(x)$. Let $x \in A \in \tilde{R}_m$ be such that $\tilde{m}_p(A) < \theta$. Let $Y = X_{p,\theta}$ and let $V \in \mathcal{R}$ be such that $V \cap Y = A \cap Y$. Since $x \in A \cap Y$, we have that $x \in V$ and so $m_p(V) \geq N_{m,p}(x) > \theta$. Let $D \in \mathcal{R}$, $D \subset V$ be such that $p(m(D)) > \theta$. Now $p(\tilde{m}(D \cap A)) \leq \tilde{m}_p(A) < \theta$ and hence

$$p(m(D)) = p(\tilde{m}(D \cap A^c)) \leq \sup_{y \in D \setminus A} N_{m,p}(y).$$

But, for $y \in D \setminus A$, we have that $N_{m,p}(y) < \theta$ since $D \subset V$ and $A \cap Y = V \cap Y$. Thus $\theta < p(m(D)) \leq \theta$, a contradiction. This completes the proof.

Lemma 5.9 *For $A \subset X$, we have $A \in \tilde{R}_m$ iff A is $\tau_{\tilde{R}_m}$-clopen.*

Proof : Clearly every $A \in \tilde{R}_m$ is $\tau_{\tilde{R}_m}$-clopen. On the other hand let A be $\tau_{\tilde{R}_m}$-clopen and let $p \in cs(E)$, $\epsilon > 0$. Since $\tau_{\mathcal{R}}$ and $\tau_{\tilde{R}_m}$ induce the same topology on $X_{p,\epsilon}$, the set $G = A \bigcap X_{p,\epsilon}$ is clopen in $X_{p,\epsilon}$ for the topology induced by $\tau_{\mathcal{R}}$. For each $x \in G$, there exists an $A_x \in \mathcal{R}$ such that $x \in A_x \bigcap X_{p,\epsilon} \subset G$. As G is $\tau_{\mathcal{R}}$-compact, there are $x_1, x_2, \ldots, x_n \in G$ such that

$$G = \bigcup_{k=1}^{n} A_{x_k} \cap X_{p,\epsilon} = V \cap X_{p,\epsilon},$$

where $V = \bigcup_{k=1}^{n} A_{x_k} \in \mathcal{R}$. In view of Theorem 5.6, A is in $\tilde{R}_m$ and the result follows.

Theorem 5.10 $\tilde{m} \in M_\tau(\tilde{R}_m, E)$.

Proof : Let $\mathcal{A}$ be a family in $\tilde{R}_m$ which decreases to the empty set and let $p \in cs(E)$, $\epsilon > 0$, $Y = X_{p,\epsilon}$. For each A in $\mathcal{A}$, there exists $B \in \mathcal{R}$ such that $B \cap Y = A \cap Y$. Let

$$\mathcal{B} = \{B \in \mathcal{R} : \exists A \in \mathcal{A}, A \cap Y = B \cap Y\}.$$

It is easy to see that $\mathcal{B} \downarrow \emptyset$. Since $m \in M_\tau(\mathcal{R}, E)$, there exists $B \in \mathcal{B}$ such that $m_p(B) < \epsilon$. Let $A \in \mathcal{A}$ be such that $A \cap Y = B \cap Y$. If $x \in A$, then $x \notin Y$ and so $N_{m,p})x) < \epsilon$. If $G \in \tilde{R}_m$ is contained in A, then

$$p(\tilde{m}(G)) \leq \sup_{x \in G} N_{m,p}(x) \leq \epsilon$$

and so $\tilde{m}_p(A) \leq \epsilon$. This proves that

$$\lim_{A \in \mathcal{A}} \tilde{m}_p(A) = 0$$

and so $\tilde{m} \in M_\tau(\tilde{R}_m, E)$.

Lemma 5.11 *If $g \in S(\tilde{R}_m)$, then for each $p \in cs(E)$ and each $\epsilon > 0$, there exists an $h \in S(\mathcal{R})$ such that $\|h - g\|_{N_{m,p}} \leq \epsilon$.*

Proof : Assume that $g \neq 0$ and let $A_1, A_2, \ldots, A_n$ be pairwise disjoint members of $\tilde{R}_m$ and non-zero scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $g = \sum_{k=1}^{n} \alpha_k \chi_{A_k}$. Let $r = \max_k |\alpha_k|$. For each k, there exists a $B_k \in \mathcal{R}$ such that $N_{m,p} < \epsilon/r$ on $A_k \triangle B_k$. Since

$$\|\alpha_k \chi_{A_k} - \alpha_k \chi_{B_k}\|_{N_{m,p}} \leq |\alpha_k| \cdot \sup_{x \in A_k \triangle B_k} N_{m,p}(x) \leq \epsilon,$$

it follows that $\|h - g\|_{N_{m,p}} \leq \epsilon$.

Using Lemmas 5.7 and 5.11, we get the following

Theorem 5.12 *A function $f \in \mathbb{K}^X$ is (VR)-integrable with respect to m iff it is (VR)-integrable with respect to $\tilde{m}$. Moreover*

$$(VR)\int f\,dm = (VR)\int f\,d\tilde{m}.$$

Theorem 5.13 *If $f \in \mathbb{K}^X$ is m-integrable with respect to m, then it is also (VR)-integrable and*

$$\int f\,dm = (VR)\int f\,dm.$$

Proof : Let $p \in cs(E)$ and $\epsilon > 0$. There exists a $\mathcal{R}$-partition $\{V_1, V_2, \ldots, V_n\}$ of X such that $|f(x) - f(y)| \cdot m_p(V_i) < \epsilon$ if $x, y \in V_i$. Let $x_k \in V_k$ and $g = \sum_{k=1}^{n} f(x_k)\chi_{V_k}$. For $x \in V_k$, we have

$$|f(x) - g(x)| \cdot N_{m,p}(x) = |f(x) - f(x_k)| \cdot N_{m,p}(x) \leq |f(x) - f(x_k)| \cdot m_p(V_k) < \epsilon.$$

This proves that f is (VR)-integrable. Also $p\left(\int f\,dm - \int g\,dm\right) \leq \epsilon$ and

$$p\left((VR)\int f\,dm - \int g\,dm\right) = p\left((VR)\int (f-g)\,dm\right) \leq \|f - g\|_{N_{m,p}} \leq \epsilon.$$

Thus

$$p\left(\int f\,dm - (VR)\int f\,dm\right) \leq \epsilon.$$

Since E is Hausdorff, it follows that

$$\int f\,dm = (VR)\int f\,dm.$$

and the proof is complete.

Theorem 5.14 *Let Y be a zero-dimensional topological space and $f : X \to Y$. Then f is $\tau_{\tilde{R}_m}$-continuous iff, for each $p \in cs(E)$ and each $\epsilon > 0$, the restriction of f to $X_{p,\epsilon}$ is $\tau_{\mathcal{R}}$- continuous.*

Proof : Since $\tau_{\mathcal{R}}$ and $\tau_{\tilde{R}_m}$ induce the same topology on $X_{p,\epsilon}$, the necessity of the condition is clear. On the other hand, assume that the condition is satisfied and let Z be a clopen subset of Y. We need to show that $f^{-1}(A)$ is $\tau_{\tilde{R}_m}$-clopen, or equivalently that $f^{-1}(A) \in \tilde{R}_m$. Let $p \in cs(E)$ and $\epsilon > 0$. The restriction h of f to $X_{p,\epsilon}$ is $\tau_{\mathcal{R}}$-continuous. Thus

$$G = f^{-1}(A) \cap X_{p,\epsilon} = h^{-1}(A)$$

is clopen in $X_{p,\epsilon}$ for the topology induced by $\tau_{\mathcal{R}}$. For each $x \in G$, there exists $V_x \in \mathcal{R}$ such that $x \in V_x \bigcap X_{p,\epsilon} \subset G$. Since G is $\tau_{\mathcal{R}}$-compact, there are $x_1, x_2, \ldots, x_n$ in G such that

$$G = \bigcup_{k=1}^{n} V_{x_k} \cap X_{p,\epsilon}.$$

If $V = \bigcup_{k=1}^{n} V_{x_k} \in \mathcal{R}$, then

$$V \cap X_{p,\epsilon} = f^{-1}(A) \cap X_{p,\epsilon}.$$

In view of Lemma 5.9, we get that $f^{-1}(A) \in \tilde{R}_m$ and we are done.

Theorem 5.15 *Let $m \in M_\tau(\mathcal{R}, E)$ and $f \in \mathbb{K}^X$. Then, f is (VR)-integrable iff :*
a) f is $\tau_{\tilde{R}_m}$-continuous.
b) For each $p \in cs(E)$ and each $\epsilon > 0$, the set

$$D = \{x : |f(x)| \cdot N_{m,p}(x) \geq \epsilon\}$$

is $\tau_{\tilde{R}_m}$-compact.

Proof : Assume that f is (VR)-integrable and let $p \in cs(E)$ and $\epsilon > 0$. There exists a sequence (g_n) in $S(\mathcal{R})$ such that $\|f - g_n\|_{N_{m,p}} \to 0$. For $x \in X_{p,\epsilon}$, we have that

$$|f(x) - g_n(x)| \leq 1/\epsilon \cdot \|f - g_n\|_{N_{m,p}} \to 0$$

uniformly. Since each g_n is $\tau_{\mathcal{R}}$-continuous , it follows that $f|_{X_{p,\epsilon}}$ is $\tau_{\mathcal{R}}$-continuous and so f is $\tau_{\tilde{R}_m}$-continuous. Also, given $\epsilon > 0$, there exists a $g \in S(\mathcal{R})$ such that $\|f - g\|_{N_{m,p}} \leq \epsilon$. Let $\{V_1, V_2, \ldots, V_n\}$ be pairwise disjoint members of $\mathcal{R}$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ non-zero scalars such that $g = \sum_{k=1}^{n} \alpha_k \chi_{V_k}$. Now

$$D = \{x : |g(x)| \cdot N_{m,p}(x) \geq \epsilon\} = \bigcup_{k=1}^{n} [V_k \cap \{x : N_{m,p}(x) \geq \epsilon/|\alpha_k|\}],$$

and so D is $\tau_{\tilde{R}_m}$-compact. Moreover

$$D = \{x : |f(x)| \cdot N_{m,p}(x) \geq \epsilon\}.$$

Conversely, assume that the conditions (a), (b) are satisfied. Let $p \in cs(E)$, $\epsilon > 0$ and

$$D = \{x : |f(x)| \cdot N_{m,p}(x) \geq \epsilon\}.$$

For each $x \in D$, there exists an $A_x \in \tilde{R}_m$ such that

$$x \in A_x \subset \{y : |f(y) - f(x)| < \epsilon/m_p(X)\}.$$

By the $\tilde{R}_m$-compactness of D, there are $y_1, y_2, \ldots, y_n \in Y$ such that $D \subset \bigcup_{k=1}^{n} A_{y_k}$. Now, there are pairwise disjoint sets $V_1, V_2, \ldots, V_N$ in $\tilde{R}_m$ such that $D \subset \bigcup_{j=1}^{N} V_j$ and each V_j is contained in some A_{y_k}. Let

$$x_j \in V_j, \quad g = \sum_{j=1}^{N} f(x_j)\chi_{V_j}.$$

If $x \in V_j$, then

$$|f(x) - g(x)| \cdot N_{m,p}(x) = |f(x) - f(x_j)| \cdot N_{m,p}(x) \leq \|m\|_p \cdot \epsilon/\|m\|_p = \epsilon,$$

while, for $x \notin \bigcup_{j=1}^{N} V_j$ we have $g(x) = 0$ and $x \notin D$, which implies that

$$|f(x) - g(x)| \cdot N_{m,p}(x) = |f(x)| \cdot N_{m,p}(x) \leq \epsilon.$$

This proves that f is (VR)-integrable with respect to $\tilde{m}$ and hence it is (VR)-integrable with respect to m. This completes the proof.

6 The Measure $\tilde{m}_f$

In this section we will assume that E is a complete Hausdorff locally convex space, $\mathcal{R}$ a separating algebra of subsets of a set X and $m \in M_\tau(\mathcal{R}, E)$. Let $f \in \mathbb{K}^X$ be (VR)-integrable with respect to m and define

$$\tilde{m}_f : \tilde{R}_m \to E, \quad \tilde{m}_f(A) = (VR)\int_A f\,dm = (VR)\int \chi_A f\,dm.$$

Then, for each $p \in cs(E)$, we have

$$p(\tilde{m}_f(A)) \le \sup_{x\in A} |f(x)| \cdot N_{m,p}(x) \le \|f\|_{N_{m,p}},$$

and so $\tilde{m}_f$ is bounded and clearly finitely-additive. Also $\tilde{m}_f$ is τ-additive. Indeed, let (A_δ) be a net in $\tilde{R}_m$ which decreases to the empty set and let $p \in cs(E)$, $\epsilon > 0$. There exists a $g \in S(\mathcal{R})$ such that $\|f - g\|_{N_{m,p}} < \epsilon$. If $g = \sum_{k=1}^n \alpha_k \chi_{V_k}$, where $V_1, V_2, \ldots, V_n$ are pairwise disjoint members of $\mathcal{R}$, then

$$\tilde{m}_g(A_\delta) = \sum_{k=1}^n \alpha_k \tilde{m}(V_k \cap A_\delta).$$

Since $A_\delta \cap V_k \downarrow \emptyset$ and $\tilde{m}$ is τ-additive, there exists δ_o such that $p(\tilde{m}_g(A_\delta)) < \epsilon$ if $\delta \ge \delta_o$. Also

$$p(\tilde{m}_{f-g}(A_\delta)) \le \|f - g\|_{N_{m,p}} < \epsilon.$$

Thus, for $\delta \ge \delta_o$, we have that $p(\tilde{m}_f(A_\delta)) < \epsilon$, which proves that $\tilde{m}_f \in M_\tau(\tilde{R}_m, E)$.

Lemma 6.1 *If $g \in S(\mathcal{R})$, then $N_{\tilde{m}_g,p}(x) = |g(x)| \cdot N_{m,p}(x)$.*

Proof : Let $g = \sum_{k=1}^n \alpha_k \chi_{V_k}$, where $\{V_1, V_2, \ldots, V_n\}$ is an $\mathcal{R}$-partition of X. Let $x \in V_k$ and $h = \alpha_k \chi_{V_k}$. If $A \in \tilde{R}_m$ is contained in V_k, then

$$\tilde{m}_g(A) = \tilde{m}_h(A) = \alpha_k \cdot (VR)\int \chi_A\,dm = g(x)\tilde{m}(A).$$

Thus

$$N_{\tilde{m}_g,p}(x) = |g(x)| \cdot N_{\tilde{m},p}(x) = |g(x)| \cdot N_{m,p}(x).$$

Lemma 6.2 *Let $f, g \in \mathbb{K}^X$ be (VR)-integrable with respect to m. Then for each $V \in \tilde{R}_m$, we have*

$$|(\tilde{m}_f)_p(V) - (\tilde{m}_g)_p(V)| \le \|f - g\|_{N_{m,p}}.$$

Proof : Assume (say) that $(\tilde{m}_f)_p(V) - (\tilde{m}_g)_p(V)| \ge 0$. Given $\epsilon > 0$, there exists $A \in \tilde{R}_m$ contained in V such that $(\tilde{m}_f)_p(V) < p(\tilde{m}_f(A)) + \epsilon$. Now

$$\begin{aligned} 0 \le (\tilde{m}_f)_p(V) - (\tilde{m}_g)_p(V) &< \epsilon + p(\tilde{m}_f(A)) - p(\tilde{m}_g(A)) \\ &\le \epsilon + p(\tilde{m}_f(A) - \tilde{m}_g(A)) \\ &= \epsilon + p(\tilde{m}_{f-g}(A)) \le \epsilon + \|f - g\|_{N_{m,p}} \end{aligned}$$

and the Lemma follows taking $\epsilon \to 0$.

Lemma 6.3 *Let $f, g \in \mathbb{K}^X$ be (VR)-integrable with respect to m. Then*

$$|N_{\tilde{m}_f,p}(x) - N_{\tilde{m}_g,p}(x)| \leq \|f - g\|_{N_{m,p}}.$$

Proof : Suppose (say) that $0 \leq N_{\tilde{m}_f,p}(x) - N_{\tilde{m}_g,p}(x)$ and choose a $V \in \tilde{R}_m$ containing x such that $(\tilde{m}_g(V) < N_{\tilde{m}_g,p}(x) + \epsilon$. Now

$$0 \leq N_{\tilde{m}_f,p}(x) - N_{\tilde{m}_g,p}(x) \leq (\tilde{m}_f)_p(V) - [(\tilde{m}_g)_p(V) - \epsilon] \leq \epsilon + \|f - g\|_{N_{m,p}}.$$

Taking $\epsilon \to 0$, the Lemma follows.

Theorem 6.4 *If $f \in \mathbb{K}^X$ is (VR)-integrable with respect to m, then*

$$N_{\tilde{m}_f,p}(x) = |f(x)| \cdot N_{m,p}(x).$$

Proof : Given $\epsilon > 0$, there exists a $g \in S(\mathcal{R})$ such that $\|f - g\|_{N_{m,p}} < \epsilon$. By Lemma 6.1, we have $N_{\tilde{m}_g,p}(x) = |g(x)| \cdot N_{m,p}(x)$. Also

$$|\,|g(x)| \cdot N_{m,p}(x) - |f(x)| \cdot N_{m,p}(x)\,| \leq |g(x) - f(x)| \cdot N_{m,p}(x) < \epsilon.$$

Thus

$$|N_{\tilde{m}_f,p} - |f(x)| \cdot N_{m,p}(x)| \leq |N_{\tilde{m}_f,p}(x) - N_{\tilde{m}_g,p}(x)| + |\,|g(x)| \cdot N_{m,p}(x) - |f(x)| \cdot N_{m,p}(x)\,| \leq 2\epsilon.$$

As $\epsilon > 0$ was arbitrary, the Theorem follows.

Lemma 6.5 *If $f \in \mathbb{K}^X$ is (VR)-integrable with respect to m and $h \in S(\mathcal{R})$, then hf is (VR)-integrable.*

Proof : Let $\epsilon > 0$, $p \in cs(E)$, $d > \|h\|$. Choose $g \in S(\mathcal{R})$ such that $\|g - f\|_{N_{m,p}} < \epsilon/d$. Now $gh \in S(\mathcal{R})$ and $\|hf - gh\|_{N_{m,p}} < \epsilon$, which proves the Lemma.

Theorem 6.6 *Let $f \in \mathbb{K}^X$ be (VR)-integrable with respect to m. If $g \in \mathbb{K}^X$ is (VR)-integrable with respect to $\tilde{m}_f$, then gf is (VR)-integrable with respect to m and*

$$(VR)\int gf\, dm = (VR)\int g\, d\tilde{m}_f.$$

Proof : Given $p \in cs(E)$ and $\epsilon > 0$,let $h \in S(\tilde{R}_m)$ be such that $\|g - h\|_{N_{\tilde{m}_f,p}} < \epsilon$. Let $d > \|h\|$ and choose $f_1 \in S(\mathcal{R})$ such that $\|f - f_1\|_{N_{m,p}} < \epsilon/d$. If $V \in \tilde{R}_m$, then

$$\int \chi_V\, d\tilde{m}_f = \tilde{m}_f(V) = (VR)\int \chi_V f\, dm$$

and so $\int h\, d\tilde{m}_f = (VR)\int hf\, dm$. Now

$$p\left((VR)\int g\, d\tilde{m}_f - \int h\, d\tilde{m}_f\right) \leq \|g - h\|_{N_{\tilde{m}_f,p}} < \epsilon.$$

If $f_2 = f - f_1$, then

$$\|hf_2\|_{N_{m,p}} \le \epsilon \quad \text{and} \quad \|g - h)f\|_{N_{m,p}} = \|g - h\|_{N_{\tilde{m}_f,p}} \le \epsilon.$$

It follows that $\|gf - hf_1\|_{N_{m,p}} \le \epsilon$. Since hf_1 is (VR)-integrable with respect to m, we get that gf is (VR)-integrable with respect to m. Also,

$$p\left((VR)\int fg\,dm - (VR)\int hf\,dm\right) \le \|gf - hf\|_{N_{m,p}} \le \epsilon.$$

It follows that

$$p\left((VR)\int fg\,dm - (VR)\int g\,d\tilde{m}_f\right) \le \epsilon,$$

which clearly completes the proof.

Theorem 6.7 *Let $f, g \in \mathbb{K}^X$ be (VR)-integrable with respect to m. If g is bounded, then :*

1. *g is (VR)-integrable with respect to $\tilde{m}_f$.*
2. *gf is (VR)-integrable with respect to m.*
3. $(VR)\int gf\,dm = (VR)\int g\,d\tilde{m}_f$.

The same result holds if we assume that f is bounded.

Proof : Assume that g is bounded. In view of the preceding Theorem, we only need to prove (1). By Theorem 5.15, g is $\tau_{\tilde{R}_m}$-continuous. As g was assumed to be bounded, we get that g is integrable with respet to $\tilde{m}_f$, which implies that it is (VR)-integrable with respect to the same measure (by Theorem 5.13). Thus (1) holds. In case f is bounded, let $d > \|f\|$ and choose $h \in S(\mathcal{R})$ such that $\|g - h\|_{N_{m,p}} < \epsilon/d$. Now

$$\|g - h\|_{N_{\tilde{m}_f,p}} = \|(g - h)f\|_{N_{m,p}} < \epsilon,$$

and so the result follows.

Theorem 6.8 *Let $f \in \mathbb{K}^X$ be (VR)-integrable with respect to m and let $g \in \mathbb{K}^X$ be m-integrable. Then :*

1. *g is (VR)-integrable with respect to $\tilde{m}_f$.*
2. *gf is (VR)-integrable with respect to m.*
3. $(VR)\int gf\,dm = (VR)\int g\,d\tilde{m}_f$.

Proof : Let $p \in cs(E)$ and $\epsilon > 0$. Since g is m-integrable, there exists a $V \in S(\mathcal{R})$, with $m_p(V^c) = 0$, such that $\|g\|_V = d < \infty$. Let $g_1 = g\chi_V$. By the preceding Theorem, there exists an $h \in S(\tilde{R}_m)$ such that $\|g_1 - h\|_{N_{\tilde{m}_f,p}} < \epsilon$. For $x \in V^c$, we have

$$|g(x) - h(x)| \cdot N_{\tilde{m}_f,p}(x) = |f(x)(g(x) - h(x))| \cdot N_{m,p}(x) = 0.$$

Thus $\|g - h\|_{N_{\tilde{m}_f,p}} \le \epsilon$. This proves (1) and the result follows.

7 The Completion of $(S(\mathcal{R}), \phi_\tau)$

In this section, $\mathcal{R}$ will be a separating algebra of subsets of a non-empty set X. We will equip X with the topology $\tau_\mathcal{R}$. As in [9], we will denote by $X^{(k)}$ the set X equipped with the zero-dimensional topology which has as a base the family of all subsets A of X such that $A \cap Y$ is clopen in Y for each compact subset Y of X. We will prove that $(C_b(X^{(k)}), \beta_o)$ coincides with the completion $\hat{F}$ of $F = (S(\mathcal{R}), \phi_\tau)$. As F is a polar Hausdorff space, its completion is the space of all linear functionals on $F' = M_\tau(\mathcal{R})$ which are $\sigma(F', F)$-continuous on ϕ_τ-equicontinuous subsets of $M_\tau(\mathcal{R})$ (see [10]). The topology of $\hat{F}$ is the one of uniform convergence on the ϕ_τ-equicontinuous subsets of $M_\tau(\mathcal{R})$. Since ϕ_τ is the topology induced on $S(\mathcal{R})$ by β_o and since β_o and the topology τ_u of uniform convergence have the same bounded sets, it follows that the strong topology on F' is the tropology given by the norm $m \mapsto \|m\|$.

Theorem 7.1 *The completion $\hat{F}$ of F is an algebraic subspace of the second dual F''. The topology of $\hat{F}$ is coarser than the topology induced on $\hat{F}$ by the norm topology of F''.*

Proof : Let u be a linear functionmal on $M_\tau(\mathcal{R})$ which is $\sigma(F', F)$-continuous on ϕ_τ-equicontinuous subsets of $M_\tau(\mathcal{R})$. Then u is norm-continuous. Indeed, let (m_n) be a sequence in $M_\tau(\mathcal{R})$ with $\|m_n\| \to 0$. The set $H = \{m_n : n \in \mathbf{N}\}$ is uniformly τ-additive. In fact, let (V_δ) be a net in $\mathcal{R}$ which decreases to the empty set and let $\epsilon > 0$. Choose n_o such that $\|m_n\| < \epsilon$ if $n > n_o$. If δ_o is such that $|m_n|(V_\delta) < \epsilon$ for all $\delta \geq \delta_o$ and all $n = 1, 2, \ldots, n_o$, then $|m|(V_\delta) < \epsilon$ for all $m \in H$ and all $\delta \geq \delta_o$. In view of Theorem 3.10, H is ϕ_τ-equicontinuous. As $\int g\, dm_n \to 0$ for all $g \in S(\mathcal{R})$, it follows that $u(m_n) \to 0$ and so $u \in F''$. The last assertion is a consequence of the fact that every ϕ_τ-equicontinuous subset of $M_\tau(\mathcal{R})$ is norm bounded.

Let $K(X)$ be the algebra of all $\tau_\mathcal{R}$- clopen subsets of X. For $m \in M_\tau(\mathcal{R}, E)$, let

$$\tilde{m} : K(X) \to \mathbb{K}, \quad \tilde{m}(A) = \int \chi_A\, dm.$$

Then $\tilde{m} \in M_\tau(K(X))$.

Lemma 7.2 *If H is a uniformly τ-additive subset of $M_\tau(\mathcal{R})$, then the set*

$$\tilde{H} = \{\tilde{m} : m \in H\}$$

is a uniformly τ-additive subset of $M_\tau(K(X))$.

Proof : Let (V_δ) be a net in $K(X)$ which decreases to the empty set. Consider the family $\mathcal{F}$ of all $A \in \mathcal{R}$ which contain some V_δ. Let $A_1, A_2 \in \mathcal{F}$ and let δ_1, δ_2 be such that $V_{\delta_i} \subset A_i$, for $i = 1, 2$. If $\delta \geq \delta_1, \delta_2$, then $V_\delta \subset A = A_1 \cap A_2$, which proves that $\mathcal{F}$ is downwards directed. Also, $\bigcap \mathcal{F} = \emptyset$. Indeed, let $x \in X$ and choose V_δ not containig x. There exists a $B \in S(\mathcal{R}$ such that $x \in B \subset V_\delta^c$. Now $V_\delta \subset A = B^c$ and $x \notin A$, which proves that $\bigcap \mathcal{F} = \emptyset$. As H is uniformly τ-additive, there exists $A \in \mathcal{F}$ with $|m|(A) < \epsilon$ for all $m \in H$. If V_δ is contained in A, then $|\tilde{m}|(V_\delta) \leq |\tilde{m}|(A) = |m|(A) < \epsilon$, for all $m \in H$, and the Lemma follows.

Theorem 7.3 *$(C_b(X), \beta_o)$ is a topological subspace of $\hat{F}$.*

Proof : Let $f \in C_b(X)$. Without loss of generality we may assume that $\|f\| \leq 1$. For each $m \in M_\tau(\mathcal{R})$, the integral $\int f\, dm$ exists. Thus f may be considered as a linear functional on $M_\tau(\mathcal{R}) = F'$. Let H be an absolutely convex ϕ_τ-equicontinuous subset of $M_\tau(\mathcal{R})$ and let (m_δ) be a net in H which is $\sigma(F', F)$-convergent to zero. We will show that $\int f\, dm_\delta \to 0$. As H is ϕ_τ-equicontinuous, we have that $d = \sup_{m\in H} \|m\| < \infty$. By the preceding Lemma, the set $\tilde{H}$ is a norm-bounded uniformly τ-additive subset of $M_\tau(K(X))$. By [4], Theorem 3.6, given $\epsilon > 0$, there exists a compact subset Y of X such that $|m|(V) = |\tilde{m}|(V) < \epsilon$ for all $m \in H$ and all $V \in \mathcal{R}$ disjoint from Y. For each $x \in Y$, there exists an $A_x \in \mathcal{R}$ containing x and such that

$$A_x \subset \{y : |f(y) - f(x)| < \epsilon/d\}.$$

By the compactness of Y, there are $x_1, x_2, \ldots, x_n$ in Y such that $Y \subset \bigcup_{k=1}^n A_{x_k}$. Now there are pairwise disjoint sets $B_1, B_2, \ldots, B_N$ in $\mathcal{R}$ covering Y such that each B_i is contained in some A_{x_k}. Let $y_i \in B_i$ and $g = \sum_{i=1}^N f(y_i)\chi_{B_i}$. For $x \in B = \bigcup_{i=1}^N B_i$, we have that $|f(x) - g(x)| < \epsilon/d$ and $|m|(B^c) < \epsilon$ for all $m \in H$. Let δ_o be such that $|\int g\, dm_\delta| < \epsilon$ if $\delta \geq \delta_o$. Since

$$\left|\int_B (f-g)\, dm_\delta\right| \leq d \cdot \|f - g\|_B \leq \epsilon \quad \text{and} \quad \left|\int_{B^c} (f-g)\, dm_\delta\right| \leq |m|(B^c) < \epsilon,$$

it follows that $|\int f\, dm_\delta| \leq \epsilon$ for all $\delta \geq \delta_o$. This proves that $f \in \hat{F}$.
Since β_o is polar, it follows from [4], Theorem 3.6, that β_o is the topology of uniform convergence on the family of all norm-bounded uniformly τ-additive subsets of $M_\tau(K(X))$. Let Z be such a subset of $M_\tau(K(X))$ and let $H = \{m|_\mathcal{R} : m \in Z\}$. Then H is uniformly τ-additive subset of $M_\tau(\mathcal{R})$ and

$$\sup_{\mu\in H} \|\mu\| = \sup_{m\in Z} \|m\| < \infty.$$

If H^o is the polar of H in $\hat{F}$ and Z^o the polar of Z in $C_b(X)$, then $Z^o = H^o \bigcap C_b(X)$. Now the result follows from this, the preceding Lemma and Theorem 3.10.

Theorem 7.4 *The completion of the space $F = (S(\mathcal{R}), \phi_\tau)$ coincides with the space $(C_b(X^{(k)}, \beta_o)$.*

Proof : By the preceding Theorem, $(C_b(X), \beta_o)$ is a topological subspace of $\hat{F}$. Thus $\hat{F}$ coincides with the completion of $(C_b(X), \beta_o)$. Now the result follows from [8], Theorem 4.3, in view of [9], Theorem 3.14

Let now E be a complete locally convex Hausdorff space and let $m \in M_\tau(\mathcal{R}, E)$. In view of the preceding Theorem, there exists a unique β_o-continuous extension u of $\hat{m}$ to all of $C_b(X^{(k)})$. We will show that, for all $f \in C_b(X^{(k)})$ we have $u(f) = (VR)\int f\, dm$.

Theorem 7.5 *Let $m \in M_\tau(\mathcal{R}, E)$, where E is a complete Hausdorff locally convex space. If*

$$u : (C_b(X^{(k)}), \beta_o) \to E$$

is the unique continuous extension of $\hat{m}$*, then* $u(f) = (VR)\int f\,dm$.

Proof : Let $f \in C_b(X^{(k)})$. Without loss of generality, we may assume that $\|f\| \le 1$. Let Γ be the set of all $\gamma = (p, Y, n)$, where $p \in cs(E)$, $n \in \mathbf{N}$ and Y a compact subset of X. We make Γ into a directed set by defining $(p_1, Y_1, n_1) \ge (p_2, Y_2, n_2)$ iff $p_1 \ge p_2$, $Y_2 \subset Y_1$ and $n_1 \ge n_2$. Let

$$B = \{g \in C_b(X^{(k)}) : \|g\| \le 1\}.$$

On B, β_o coincides with the topology of uniform convergence on the compact subsets of $X^{(k)}$ (equivalently on compact subsets of X by [9], Corollary 3.14).
Claim: For each $\gamma = (p, Y, n)$, there exists a $g_\gamma \in S(\mathcal{R})$, $g_\gamma \in B$, such that

$$\|f - g_\gamma\|_Y \le 1/n, \quad \|f - g_\gamma\|_{N_{m,p}} \le 1/n.$$

Indeed, choose $\epsilon > 0$ such that $\epsilon < 1/n$ and $\epsilon \cdot \|m\|_p < 1/n$. The set

$$Z = Y \bigcup \{x : N_{m,p}(x) \ge \epsilon\}$$

is compact. For each $y \in Z$, there exists $V_y \in S(\mathcal{R})$ containing y and such that

$$V_y \cap Z \subset \{z : |f(z) - f(y)| < \epsilon\}.$$

By the compactness of Z, there are pairwise disjoint $W_1, W_2, \ldots, W_N$ in $S(\mathcal{R})$ covering Z and such that each W_i is contained in some V_y. Choose $z_k \in W_k$ and take $g_\gamma = \sum_{k=1}^N f(z_k)\chi_{W_k}$. Then $g_\gamma \in B$. If $x \in Y$, then $|f(x) - g_\gamma(x)| \le \epsilon < 1/n$ and so $\|f - g_\gamma\|_Y \le 1/n$. Also, if $x \in W = \bigcup_{k=1}^N W_k$, then

$$|f(x) - g_\gamma(x)| \cdot N_{m,p}(x) \le \epsilon \cdot \|m\|_p < 1/n,$$

while for $x \notin W$ we have that $N_{m,p}(x) \le \epsilon < 1/n$. Thus $\|f - g_\gamma\|_{N_{m,p}} \le 1/n$, which proves our claim.
Now the net (g_γ) is in B and converges to f with respect to the topology of uniform convergence on compact subsets of X and so (g_γ) is β_o-convergent to f, which implies that $u(f) = \lim u(g_\gamma)$. On the other hand, (g_γ) is contained in G_m and converges to f in the topology of G_m Thus

$$u(f) = \lim u(g_\gamma) = \lim \int g_\gamma\,dm = (VR)\int f\,dm.$$

This completes the proof.

Theorem 7.6 *Let* X *be a zero-dimensional Hausdorff space and let* Δ *be the family of all pairs* (m, p) *for which there exists a Hausdorff locally convex space* E *such that* $p \in cs(E)$ *and* $m \in M_\tau(K(X), E)$*, where* $K(X)$ *is the algebra of all clopen subsets of* X*. Then the topologies* β *and* β_o *on* $C_b(X)$ *coincide with the locally convex topology* ρ *generated by the seminorms* $\|\cdot\|_{N_{m,p}}$, $(m, p) \in \Delta$.

Proof : As it is shown in the proof of the preceding Theorem, the space $F = S(K(X))$ is ρ-dense in $C_b(X)$. Also F is dense in $C_b(X)$ for the topologies β and β_o. In view of Theorems 4.15, 4.18 and 4.19, the topologies β_o, β and ρ coincide on F. Also, ρ is coarser than β_o. Indeed, let $(m,p) \in \Delta$ and

$$V = \{f \in C_b(X) : \|f\|_{N_{m,p}} \leq 1\}.$$

Let $r > 0$ and choose $0 < \epsilon < 1/r$ such that $\epsilon \cdot m_p(X) < 1$. The set $Y = \{x : N_{m,p}(X) \geq \epsilon\}$ is compact. Moreover

$$V_1 = \{f \in C_b(X) : \|f\| \leq r, \ \|f\|_Y \leq \epsilon\}$$

is contained in V. In fact, let $f \in V_1$. If $x \in Y$, then $|f(x)| \cdot N_{m,p}(x) \leq \epsilon \cdot m_p(X) \leq 1$, while for $x \notin Y$ we have $|f(x)| \cdot N_{m,p}(x) \leq r\epsilon \leq 1$. Thus $\|f\|_{N_{m,p}} \leq 1$, i.e. $f \in V$. This, being true for each $r > 0$, implies that V is a β_o-neighborhood of zero. Now the result follows from Lemma 4.17.

References

[1] J. Aguayo, Vector measures and integral operators, in : Ultrametric Functional Analysis, Cont. Math., vol. **384**(2005), 1-13.

[2] J. Aguayo and T. E. Gilsdorf, Non-Archimedean vector measures and integral operators, in : p-adic Functional Analysis, Lecture Notes in Pure and Appplied Mathematics, vol **222**, Marcel Dekker, New York (2001), 1-11.

[3] J. Aguayo and M Nova, Non-Archimedean integral operators on the space of continuous functions, in : Ultrametric Functional Analysis, Cont. Math., vol **319**(2002), 1-15.

[4] A. K. Katsaras, The strict topology in non-Archimedean vector-valued function spaces, Proc. Kon. Ned. Akad. Wet. A **87** **(2)** (1984), 189-201.

[5] A. K. Katsaras, Strict topologies in non-Archimedean function spaces, Intern. J. Math. and Math. Sci., **7** **(1)**, (1984), 23-33.

[6] A. K. Katsaras, Separable measures and strict topologies on spaces of non-Archimedean valued functions, in : P-adic Numbers in Number Theory, Analytic Geometry and Functional Analysis, edided by S. Caenepeel, Bull. Belgian Math., (2002), 117-139.

[7] A. K. Katsaras, Strict topologies and vector measures on non-Archimedean spaces, Cont. Math. vol. **319** (2003), 109-129.

[8] A. K. Katsaras, Non-Archimedean integration and strict topologies, Cont. Math., vol. **384** (2005), 111-144.

[9] A. K. Katsaras, p-adic measures and p-adic spaces of continuous functions (preprint).

[10] A. K. Katsaras, The non-Archimedean Grothendieck's completeness theorem, Bull. Inst. Math. Acad. Sinica **19**(1991), 351-354.

[11] A. F. Monna and T. A. Springer, Integration non-archimedienne, Indag. Math. **25, no 4**(1963), 634-653.

[12] W. H. Schikhof, Locally convex spaces over non-spherically complete fields I, II, Bull. Soc. Math. Belg., Ser. B, **38** (1986), 187-224.

[13] A. C. M. van Rooij, Non-Archimedean Functional Analysis, New York and Bassel, Marcel Dekker, 1978.

[14] A. C. M. van Rooij and W. H. Schikhof, Non-Archimedean Integration Theory, Indag. Math., **31**(1969), 190-199.

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